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Indifference pricing for CRRA utilities

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Abstract We study utility indifference pricing of claim streams with intertemporal consumption and constant relative risk aversion utilities. We derive explicit formulas for the derivatives of the utility indifference price with respect to claims and wealth. The elegant structure of these formulas is a reflection of surprising algebraic identities for the derivatives of the optimal consumption stream. Namely, the partial derivative of the optimal consumption stream with respect to the endowment is always a projection. Furthermore, it is an orthogonal projection with respect to a natural "economic inner product". These algebraic identities generate cancellations between the terms entering derivatives of the indifference price and allow us to prove sharp global bounds for the indifference price that become exact when the claims to wealth ratio is large and risk aversion is between one and two. For general risk aversion, we show that, in the large claims to wealth ratio limit, the asymptotic expansion of the indifference price is given in terms of fractional powers of the wealth, depending on risk aversion. When risk aversion is equal to one, the fractional power depends on the underlying claim.

Keywords Indifference pricing · Incomplete markets · Large risks

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1 Introduction

Imagine that we must either

(1) as a representative of a private bank, or, hedge fund, determine the price of an over the counter derivative (option) contract,
or
(2) determine, as a representative of a corporation, the correct price for a real option (a capital investment),
or
(3) determine, as a representative of an insurance (reinsurance) company, the price of an insurance contract.

In each case, we want to determine the optimal price of a financial contract market consistently, by exploiting correlations between the payoff of our investment and the stock market (see, Wüthrich et al. [43] for an applied theory of market consistent pricing of insurance contracts).

For a *complete* market, with asset prices following geometric Brownian motions, Black and Scholes [7] solved this pricing problem. Completeness is an essential hypothesis for their solution, because it implies that the payoff of any option can be perfectly replicated by a suitable trading strategy in stocks and bonds. The Black-Scholes price of an option is determined by arbitrage (*linearly*), and is equal to the price of the replicating strategy.

On the other hand, many aspects of the market consistent pricing problem for *incomplete* markets are still under investigation, because now, the general payoff cannot be replicated by trading in stocks and bonds.

A common dodge around market consistent pricing is to decompose the payoff (or, synonymously, contingent claim) into hedgeable and unhedgeable components. The first component is priced by arbitrage, and the second *non linearly*. In the context of insurance, the non linear component is referred to as the insurance loading, namely, the risk premium that the insured pays to the insurer. The loading depends on the risk aversion of the insurance company.

If an insurance company is "sufficiently isolated" from the financial markets, then there is a well known theoretical principle that guides the pricing of its insurance contracts. Namely, the principle of utility indifference. That is, the price of the contract is chosen so that the utility of the company is the same before and after the contract is sold. Here, we imagine that the company acts as a rational agent, maximizing a von Neumann–Morgenstern utility.

If, by contrast, the company is not isolated from the financial markets, an important modification of the basic utility indifference principle is required. Now, the company can modify its effective claims stream by choosing an appropriate trading strategy in available securities. Of course, a rational company will choose a strategy that maximizes its utility. In this context, the von Neumann–Morgenstern utility function should be replaced by the maximal utility achievable by trading. That is, the maximal utility achievable by trading is the same before and after contracts are sold.

Pricing by maximal utility achievable by trading indifference is market consistent in the sense that perfectly hedgeable contingent claims are automatically priced by arbitrage.

The same economic reasoning applies verbatim to the pricing problems (1) and (2) from above.

Hodges and Neuberger [21] and Davis [11] were the first to consider utility indifference pricing in incomplete markets. Since then, the interest to this topic has grown dramatically. See, e.g., [1–4,8,12,16–19,24,27,28,31–33,35,36,39–41]. See, also, [5,20] for a general survey of existing literature on this topic.

A necessary prerequisite for implementing the maximal utility achievable by trading indifference principle is a solution to the utility maximization problem in the presence of an unhedgeable random endowment. This important problem has been extensively studied in numerous papers. General (but nonconstructive) existence/uniqueness results (see, e.g., Cvitanic et al. [10], Karatzas and Zitkovic [25], Hugonnier and Kramkov [23] and Biagini and Frittelli [6]) and several special cases, analyzed explicitly (see, e.g., [17,18]). However, we still know very little about the general structure of optimal consumption/wealth in incomplete markets, and this is a major obstacle for gaining a deeper understanding of indifference prices.

Given that the case without a random endowment is relatively well understood, it is natural to consider the case when the random endowment is present, but is sufficiently small and try obtaining approximate expressions for the indifference prices. For the case of utility only terminal wealth (without intermediate consumption), this problem has been recently solved in two seminal papers [27,28] by Kramkov and Sirbu. Namely, they calculated explicitly indifference prices and corresponding optimal hedges in a general, semi-martingale setting with general utilities of terminal wealth when the size of the unhedgeable claim is small. In particular, they showed that the behavior of the indifference prices depends crucially on a new, remarkable property that they call “risk tolerance wealth process.”

Still, very little is known about the behavior of the indifference prices when the claims size is large. The only exception is the special case of exponential (constant absolute risk aversion, also known as CARA) utilities (see, e.g., [22,31]). The natural translation invariance of exponential utility implies that the indifference price does not depend on the agent’s wealth, which greatly facilitates the analysis.[1] For this reason, almost all existing literature on indifference prices consider the case of exponential utilities. See, e.g., [2–4,8,18,31,35–38,40,41].

This independence of exponential indifference prices on the wealth/capital is of course counterfactual in many economic situations, as already emphasized by El Karoui and Rouge [40]. For this reason, it is important to study utility indifference pricing for the case when the underlying utility has a constant relative risk aversion (the so-called isoelastic utility $u(c) = c^{1-\gamma}/(1-\gamma)$.[2] This precisely the goal of this paper.

In order to calculate the indifference price, we first explicitly construct the corresponding optimal consumption stream for a large class of incomplete markets, that we refer to as the class $\mathfrak{C}$. A key ingredient to our construction is the recursive structure of the utility maximization problem arising in this special class of incomplete markets, introduced by Malamud and Trubowitz [29]. They showed the class $\mathfrak{C}$ is characterized by several important mathematical and economic properties. For example, it is the only class for which the crucial economic properties of precautionary savings and diminishing marginal propensity to consume hold. The class $\mathfrak{C}$ also includes all discrete time diffusion driven incomplete markets and, consequently, most of the classical discrete time incomplete market models. A related recursive

[1] As an illustration consider the simple case of “tradeless” indifference pricing. The “tradeless” indifference price π of a claim y is the solution to

$$u(x) = E[u(x+\pi-y)] \tag{1.1}$$

where x is the wealth of the insurance company. If $u(x) = -e^{-\gamma x}$ then

$$\pi = \gamma^{-1} \log E[e^{\gamma y}] \tag{1.2}$$

is clearly independent of x.

[2] This utility has become a benchmark model in financial economics due to its scale-invariance properties.

structure was independently discovered by Musiela and Zariphopoulou [34–36] in the case of incomplete binomial models (see, also [37,38,42] for further developements of these ideas).

In [29], Malamud and Trubowitz exploited a local, recursive procedure for constructing optimal consumption streams. Here, we introduce (see, Proposition 4.7) a global construction. One important consequence of our new construction is that the *derivative* of the optimal consumption stream with respect to the endowment is a *projection at any point* in the space of endowments (see, Theorem 4.9). This is surprising. The projections are not immediately orthogonal. However, there is an economically natural inner product, for which the projections become orthogonal. The inner product depends on the point in the space of endowments and thus introduces a natural Riemannian structure into the model.

The projection property of the derivative makes it possible to calculate the first, second and third derivatives of the indifference price in a useful, explicit form. This property generates algebraic identities and unexpected cancellations, that allow us to obtain sharp, global bounds on the indifference prices (see, Theorem 3.4). Theorem 3.4 is the main non-asymptotic result of our paper.

We also study asymptotic behavior of the indifference prices when the claims size (or, volume, as in Becherer [2]) is large. As has been shown by Becherer [2] in the case of a constant absolute risk aversion (CARA) utility and a general semi-martingale model, the indifference price per unit of claim converges to the upper hedging price when the claims size (volume) goes to infinity. This result also holds in the setting of the current paper. However, by contrast to the CARA case, the indifference price depends on level of capital and this dependence is quite complicated. We use singular perturbation methods to expand the indifference price in capital and explicitly calculate the leading term of this expansion. Surprisingly, this term turns out to be proportional to a fractional power of capital. This fractional power (see, Theorem 3.3) depends in a very non-trivial way on both risk aversion and fixed claims. The difference between the sharp, global bounds of Theorem 3.4 and the first two terms of the small capital/large claims volume expansion goes to zero at an explicitly calculable rate.

Organization of the paper:

In Sect. 2, we introduce the class $\mathfrak{C}$ of incomplete markets, study its basic properties and define the utility maximization problem and the utility indifference price of a stream of claims.

In Sect. 3, we formulate the main results of the paper.

In Sect. 4, we construct the solution to the utility maximization problem through an explicit non-linear map and study its properties. In particular, we show that the derivative of the optimal consumption with respect to the endowment is always a projection.

In Appendix, we establish sharp, global bounds for the indifference price.

Finally, in the Appendix we present the proof of Theorem 3.3. To provide a link with Kramkov and Sirbu [27], we also calculate the second order expansion of the indifference price when the claim size is small.

2 General incomplete markets

2.1 The structure of market incompleteness

The randomness in our model is described by a **finite**, filtered probability space $(\Omega, \mathcal{G}, \mathcal{B}, P)$ where the filtration $\mathcal{G} = (\mathcal{G}_t)_{t=0}^T$ satisfies

$$\{\emptyset, \Omega\} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_T = \mathcal{B}. \tag{2.1}$$

There are T time periods. We emphasize that everywhere in this paper the probability space Ω and time horizon T are assumed to be finite. The problem of passing to a continuous time limit is a topic of ongoing research.

Definition 2.1 A financial asset is a pair of $\mathcal{G}$-adapted processes, a price process $\mathbf{p} = (p_t, t = 0, \ldots, T)$ and a dividend process $\mathbf{d} = (d_t, t = 0, \ldots, T)$.

A financial market $(\mathcal{M}, \mathcal{G})$ is a collection of financial assets.

A τ-period risk free bond at time t is the asset, whose dividend process $d_\theta = 1$ for $\theta = t + \tau$ and $d_\theta = 0$ otherwise.

For the sake of brevity, we will often use (p_t) to denote a process without indicating that $t = 0, \ldots, T$.

We allow for an arbitrary type of market incompleteness, except for a natural

Assumption 1 One period risk free bonds are available for trading at each moment of time.

Definition 2.2 Let $(\mathcal{M}, \mathcal{G}) = \{A_1, \ldots, A_N\}$ be the underlying financial market with financial assets $A_1, \ldots, A_N$. Asset A_i has a price process $(p_{i\,t})$ and a dividend process $(d_{i\,t})$. The payoff subspace $\mathcal{L}_t$ at time t is defined by

$$\mathcal{L}_t = \left\{ \sum_{i=1}^{N} x_{i\,t-1}\,(p_{i\,t} + d_{i\,t}) \mid x_{i\,t-1} \in L_2(\mathcal{G}_{t-1}) \quad \text{for all} \quad i = 1, \ldots, N \right\}. \tag{2.2}$$

This is the set of payoffs at time t of all possible $\mathcal{G}_{t-1}$-measurable investments $x_{i\,t-1}$ at time $t-1$. We denote by $\mathrm{P}_{\mathcal{L}_t}$ the orthogonal projection onto the subspace $\mathcal{L}_t$ in the space $L_2(\mathcal{G}_t)$. Similarly, let $\mathrm{P}_{\mathcal{G}_t}$, $t = 1, \ldots, T$, be the orthogonal projection (conditional expectation) from $L_2(\Omega, \mathcal{B})$ onto $L_2(\Omega, \mathcal{G}_t)$. We write $\mathbf{P}_{\mathcal{G}}$ for the orthogonal sum

$$\mathbf{P}_{\mathcal{G}} = \oplus_{t=1}^{T}\, \mathrm{P}_{\mathcal{G}_t}. \tag{2.3}$$

Note that, since the probability space is finite, all spaces $L_p(\Omega)$ coincide and are isomorphic to $L_0(\Omega)$, the space of all finite valued random variables. But, since we constantly use the Hilbert space structure in $L_2(\Omega)$ and the fact that Q_τ, see (4.15), are orthogonal projections with respect to this Hilbert space structure, we will use the notation $L_2(\Omega)$.

By Assumption 1, $1 \in L_2(\mathcal{G}_{t-1}) \subset \mathcal{L}_t$ and, consequently, $\mathrm{P}_{\mathcal{L}_t} 1 = 1$. Furthermore, for any $\mathcal{G}_{t-1}$-measurable Y and any $\mathcal{G}_t$-measurable X we have

$$\mathrm{P}_{\mathcal{L}_t}\,(X\,Y) = Y\,\mathrm{P}_{\mathcal{L}_t}\,X. \tag{2.4}$$

A portfolio strategy for an agent, with a $\mathcal{G}$-adapted individual endowment process, trading on the market $(\mathcal{M}, \mathcal{G})$ is an N dimensional, $\mathcal{G}$-adapted process $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$. Here, $\mathbf{x}_j = (x_{j0}, \ldots, x_{j\,T-1}, 0)$. The random variable $x_{j\,t}$ counts the number of shares of asset A_j held at time $t+1$ before dividends are paid and assets are traded. The last component 0 formalizes the convention that no investments are made at the final time period T.

Definition 2.3 The dividend process $\mathbf{D}_{\mathbf{x}}$ generated by the portfolio strategy $\mathbf{x}$ is

$$D_{\mathbf{x},t} = \sum_{i=1}^{N} (d_{i\,t} + p_{i\,t})\, x_{i\,t-1} - \sum_{i=1}^{N} p_{i\,t}\, x_{i\,t} \tag{2.5}$$

for $t = 0, \ldots, T$, where $\mathbf{d}_i$ and $\mathbf{p}_i$ are the dividend and price processes of the asset A_i. In particular, the initial investment is $D_{\mathbf{x},0} = -\sum_{i=1}^{N} p_{i\,0}\, x_{i\,0}$.

The process

$$X_t = X_t(\mathbf{x}) = \sum_{i=1}^{N} (d_{i\,t} + p_{i\,t})\, x_{i\,t-1} \tag{2.6}$$

is referred to as the wealth process of the strategy $\mathbf{x}$.

Definition 2.4 A market $(\mathcal{M}, \mathcal{G})$ is arbitrage free if there is no portfolio strategy $\mathbf{x}$ such that $D_{\mathbf{x},\,t} \geq 0$ for all $t = 0, \ldots, T$ and $D_{\mathbf{x},\tau} > 0$ for some τ with positive probability.

A market $(\mathcal{M}, \mathcal{G})$ is dynamically complete if for any $\mathcal{G}$-adapted process $(Y_t,\ t = 1, \ldots, T)$ there exists a portfolio strategy $\mathbf{x}$ such that

$$D_{\mathbf{x},\,t} = Y_t \tag{2.7}$$

for all $t = 1, \ldots, T$.

Definition 2.5 A $\mathcal{G}$-adapted process $\mathbf{R} = (R_t)$ is referred to as a state price density process (SPD process) for the market $(\mathcal{M}, \mathcal{G})$ if the identity

$$R_t\, p_{i\,t} = E\left[R_{t+1}\left(p_{i\,t+1} + d_{i\,t+1}\right) \middle| \mathcal{G}_t\right] \tag{2.8}$$

holds for any asset $A_i,\ i = 1, \ldots, N$, and any $t = 0, \ldots, T-1$.

In particular, under the standard no-bubble condition $p_{i\,T} = 0$, the price

$$p_{i\,t} = R_t^{-1}\, E\left[\sum_{\tau=1}^{T-\tau} R_{t+\tau}\, d_{i\,t+\tau} \middle| \mathcal{G}_t\right] \tag{2.9}$$

is the discounted value of future dividends.

The following lemma summarizes some well known properties of state price densities.

Lemma 2.6 *A market $(\mathcal{M}, \mathcal{G})$ is arbitrage free if and only if there exists a positive SPD process.*

An arbitrage free market $(\mathcal{M}, \mathcal{G})$ is dynamically complete if and only if there exists a unique, positive SPD process.

A process **D** *is a dividend process of a portfolio strategy if and only if it is orthogonal to any SPD process, i.e.,*

$$E\left[\sum_{t=0}^{T} D_t\, R_t\right] = 0 \tag{2.10}$$

for any SPD process **R**.

See, e.g., [13].

When markets are incomplete, there are infinitely many state price density processes. This is one of the main difficulties in the analysis of utility maximization in incomplete markets. Malamud and Trubowitz [29] introduced a unique, natural, "aggregate" state price density process and showed that all budget constraints and first order conditions can be formulated in terms of this special SPD process.

Lemma 2.7 *Under the assumption of no arbitrage, there exists a unique, aggregate state price density process* $\mathbf{M} = (M_t)$ *such that* $M_t \in \mathcal{L}_t$ *for all* $t = 1, \ldots, T$*. Furthermore, a process* $\mathbf{R} = (R_t)$ *is a state price density process if and only if*

$$\mathrm{P}_{\mathcal{L}_t} \frac{R_t}{R_{t-1}} = \frac{M_t}{M_{t-1}} \tag{2.11}$$

for all t*.*

See, [29, Lemma 2.5].

The aggregate SPD process $\mathbf{M}$ is natural because it lives in the market subspace, just like the prices themselves. Note that, in general, $\mathbf{M}$ is **not positive**.[3] The main source of problems is that the projection $\mathrm{P}_{\mathcal{L}_t}$ is **not** necessarily **positivity preserving**. This fact motivated the introduction of a new class $\mathfrak{C}$ of incomplete markets in Malamud and Trubowitz [29] .

Definition 2.8 An incomplete market $(\mathcal{M}, \mathcal{G})$ belongs to the class $\mathfrak{C}$ if there exists a subfiltration $\mathcal{H} = (\mathcal{H}_t,\ t = 0, \ldots, T)$ of $\mathcal{G}$ such that

- $\mathcal{H}_{t+1} \supset \mathcal{G}_t \supset \mathcal{H}_t$ for all t.
- The payoff process $(p_{i\,t} + d_{i\,t})$ of any asset A_i is adapted to $\mathcal{H}$.
- Any $\mathcal{H}_t$ measurable claim Y can be replicated by a $\mathcal{G}_{t-1}$ measurable portfolio $x_1, \ldots, x_N$ of assets, purchased at time $t-1$. That is,

$$Y = \sum_{i=1}^{N} x_i \left(p_{i\,t} + d_{i\,t}\right). \tag{2.12}$$

Equivalently, $\mathcal{L}_t = L_2(\mathcal{H}_t)$ and $\mathrm{P}_{\mathcal{L}_t} = \mathrm{P}_{\mathcal{H}_t} = E\,[\cdot\,|\,\mathcal{H}_t]$.

We refer to $\mathcal{H}$ as the hedgeable filtration.

It is possible to show (see, [29]) that $\mathrm{P}_{\mathcal{L}_t}$ is positivity preserving if and only if there exists a subalgebra $\mathcal{H}_t \subset \mathcal{G}_t$ such that $\mathrm{P}_{\mathcal{L}_t}$ is the conditional expectation relative to $\mathcal{H}_t$. In particular, it is possible to show (see, [29, Proposition 3.4]) that the aggregate state price density process $\mathbf{M}$ is the unique *positive* state price density process adapted to $\mathcal{H}$.

The class $\mathfrak{C}$ has many interesting properties. As an illustration, we present a natural subclass of incomplete markets from the class $\mathfrak{C}$.

Example 2.9 Let $\mathcal{G}$ be an arbitrary filtration on a finite probability space Ω. Consider a market, consisting of assets without dividends, for which the price processes satisfy the discrete time SDE of the form

$$p_{i\,t} = p_{i\,t-1} + \mu_{i\,t-1} + \sum_{j=1}^{N} \sigma_{i\,j\,t-1}\, X_{i\,t}, \tag{2.13}$$

where $(\sigma_{i\,j\,t}),\ (\mu_{i\,t})$ are arbitrary, $\mathcal{G}$-adapted processes and the process

$$B_{i\,\tau} = \sum_{t=0}^{\tau} X_{i\,t}$$

[3] It is possible that $M_t = 0$ with positive probability. But, in [29], $\mathbf{M}$ is constructed as a product one period stochastic discount factors M_t/M_{t-1} which are well defined.

is a martingale with respect to $\mathcal{G}$ for any $i = 1, \dots, N$. SDE (2.13) is a discrete analog of the continuous time SDE

$$dp_{i\,t} = \mu_{i\,t}\,dt + \sum_{j=1}^{N} \sigma_{i\,j\,t}\,dB_{j\,t} \tag{2.14}$$

with predictable processes $(\mu_{i\,t})$ and $(\sigma_{i\,j\,t})$. Let $\mathcal{F}$ be the natural filtration, generated by the martingales B_i, $i = 1, \dots, N$ (i.e., the minimal filtration for which the martingales are adapted). Suppose now that the martingales B_i, $i = 1, \dots, N$, have the spanning property: any $\mathcal{F}$-martingale Z_t can be represented as a stochastic integral w.r.t. B_is:

$$dZ_t = Z_t - Z_{t-1} = \sum_{i=1}^{N} \xi_{i\,t-1}\,X_{i\,t}.$$

Then, if the matrix $(\sigma_{i\,j\,t})_{i,j=1,\dots,n}$ is invertible, the market is in the class $\mathfrak{C}$ and the hedgeable σ-algebra $\mathcal{H}_t$ is given by $\mathcal{H}_t = \sigma(\mathcal{F}_t,\, \mathcal{G}_{t-1})$, the minimal algebra, generated by $\mathcal{F}_t$ and $\mathcal{G}_{t-1}$.

Recall that in a standard, diffusion driven incomplete market, price processes follow (2.14), but with B_i being Brownian motions. It is possible to show that any diffusion driven incomplete market can be approximated by a discrete time incomplete market of the above form. Here, it is important that Brownian motions naturally have the spanning property. See, [29, Sect. 4.2].

In the sequel, we make the following

Assumption 2 The market $(\mathcal{M},\, \mathcal{G})$ belongs to the class $\mathfrak{C}$. The corresponding hedgeable filtration is denoted by $\mathcal{H}$. The unique, positive, aggregate state price density process adapted to $\mathcal{H}$ and normalized by $M_0 = 1$ is denoted by $\mathbf{M} = (M_t)$.

Assumption 2 implies that the following is true.

Proposition 2.10 *For any $\mathcal{H}$-adapted process (X_t) there exists a $\mathcal{G}$-adapted portfolio strategy $\mathbf{x}$ such that (X_t) coincides with the wealth process of this strategy, $X_t = X_t(\mathbf{x})$ for all t. In this case,*

$$D_{\mathbf{x},\,t} = X_t - E\left[\frac{M_{t+1}}{M_t}\,X_{t+1}\,|\,\mathcal{G}_t\right] \tag{2.15}$$

for all t. Here, we use the standard convention $X_0 = X_{T+1} = 0$.

See, Lemma 3.4 in [30].

2.2 The budget set

Initial value of a process (stream) will play a special role in our considerations. For this reason, starting from this section, we will always treat the value of a random process at time zero separately and write a process $(w_t,\, t = 0, \dots, T)$ as $(w_0,\, \mathbf{w})$ where $\mathbf{w} = (w_t,\, t = 1, \dots, T)$.

Definition 2.11 Consider an agent endowed with an (income) stream $(w_0,\, \mathbf{w})$ of consumption good, trading in the financial market to achieve a desirable consumption stream $(c_0,\, \mathbf{c})$. A consumption stream $(c_0,\, \mathbf{c})$ is achievable by trading if there exists a $\mathcal{G}$-adapted portfolio stratetegy $\mathbf{x}$ such that

$$c_t = w_t + D_{\mathbf{x},\,t} \tag{2.16}$$

for all $t = 0, \dots, T$.

Definition 2.12 The budget set $\mathrm{B}(w_0, \mathbf{w})$ of an agent with a $\mathcal{G}$-adapted endowment process $(w_0, \mathbf{w})$ with $\mathbf{w} = (w_t, t \geq 1)$ is the set of all positive consumption streams, that can be achieved by trading.

By Proposition 2.10, the following is true

Lemma 2.13 *A stream $(c_0, \mathbf{c}) \in B(w_0, \mathbf{w})$ if and only if there exists an $\mathcal{H}$-adapted wealth process (X_t) such that*

$$c_t = w_t + X_t - E\left[\frac{M_{t+1}}{M_t} X_{t+1} \,\middle|\, \mathcal{G}_t\right] > 0 \tag{2.17}$$

for all $t = 0, \dots, T$.

In applications to pricing insurance claims, we will be in the situation when the endowment stream (w_t) takes negative values. Thus, we must make sure that the budget set is non-empty.

Definition 2.14 Let $\mathbf{Y} = (Y_t, t = 1, \dots, T)$ be a $\mathcal{G}$-adapted process. The upper hedging price for $\mathbf{Y}$ at time zero is the minimal number $\mathbf{Y}_0^u \in \mathbb{R}$ such that there exists a portfolio strategy $\mathbf{x}$ satisfying

$$D_{\mathbf{x},t} \geq Y_t \tag{2.18}$$

for all $t = 1, \dots, T$ with $-D_{\mathbf{x},0} \leq \mathbf{Y}_0^u$. A portfolio strategy $\mathbf{x}$ is called upper hedging for $\mathbf{Y}$ if (2.18) is satisfied.

In general, the calculation of the upper hedging price is a non-trivial problem. But, for incomplete markets in the class $\mathfrak{C}$, the upper hedging price can be explicitly calculated by a simple, recursive procedure.

Definition 2.15 Let $\mathfrak{A} \subset \mathcal{B}$ be a sub-σ-algebra and $X \in L^\infty(\mathcal{B})$. Let

$$\operatorname{esssup}[X \,|\, \mathfrak{A}] = \operatorname{essinf}\{Z \in L^0(\mathfrak{A}) : Z \geq X\}$$

be the conditional supremum of X relative to $\mathfrak{A}$.

When the probability space is finite, we have

$$\operatorname{esssup}[X \,|\, \mathfrak{A}] = \max[X \,|\, \mathfrak{A}]. \tag{2.19}$$

The following proposition is a direct consequence of the results of [15] (see Sect. 7.3) applied to a market in the class $\mathfrak{C}$. Nevertheless, we present a proof for the reader's convenience.

Proposition 2.16 *Let $\mathbf{Y}_{T+1}^u = 0$ and define inductively for $t \leq T$*

$$\mathbf{Y}_t^u = \max\left[Y_t + M_t^{-1}\, E\,[\mathbf{Y}_{t+1}^u\, M_{t+1} \,|\, \mathcal{G}_t] \,|\, \mathcal{H}_t\right] \tag{2.20}$$

for $t \geq 1$ and

$$\mathbf{Y}_0^u = E[M_1\, \mathbf{Y}_1].$$

Then, $\mathbf{Y}_0^u$ is the upper hedging price for the stream $\mathbf{Y}$. Furthermore, for any upper hedging strategy $\mathbf{x}$ for the stream $\mathbf{Y}$, the wealth process $X_t(\mathbf{x})$ satisfies

$$X_t(\mathbf{x}) \geq \mathbf{Y}_t^u \tag{2.21}$$

for all $t \geq 1$. Thus, $(\mathbf{Y}_t^u)$ is the minimal upper hedging wealth process. In particular, if $\mathbf{x}$ is an upper hedging strategy and $D_{\mathbf{x}\,0} = -\mathbf{Y}_0^u$ then $X_t(\mathbf{x}) = \mathbf{Y}_t^u$ for all $t = 1, \dots, T$.

Proof We do the proof by backward induction. The claim is obvious for $t = T + 1$ (we use the convention $Y_{t+1} = 0$). Suppose that the claim is proved for all $t \geq \tau + 1$ and let us prove it for $t = \tau$. We have

$$D_{\mathbf{x}\,t} = X_t - M_t^{-1}\, E[M_{t+1}\, X_{t+1}\,|\,\mathcal{G}_t] \geq Y_t \tag{2.22}$$

if and only if

$$\begin{aligned} X_t &\geq \max\left[Y_t + M_t^{-1}\, E\,[X_{t+1}\, M_{t+1}\,|\,\mathcal{G}_t]\,|\,\mathcal{H}_t\,\right] \\ &\geq \max\left[Y_t + M_t^{-1}\, E\,[\mathbf{Y}_{t+1}^u\, M_{t+1}\,|\,\mathcal{G}_t]\,|\,\mathcal{H}_t\,\right] \\ &= \mathbf{Y}_t^u \end{aligned} \tag{2.23}$$

and $X_t = \mathbf{Y}_t^u$ implies $X_{t+1} = \mathbf{Y}_{t+1}^u$. Therefore, $X_0 = \mathbf{Y}_0^u$ if and only if $X_t = \mathbf{Y}_t^u$ for all t. The proof is complete. □

Lemma 2.17 *The budget set* $B(w_0,\ \mathbf{w})$ *is non-empty if and only if*

$$w_0 \ > \ (-\mathbf{w})_0^u \tag{2.24}$$

In the sequel, we always assume that (2.24) holds.

2.3 Utility maximization problem

Consider an agent with a $\mathcal{G}$-adapted endowment process $(w_0,\ \mathbf{w})$. It is standard in the modern literature to assume that the rational behavior of the agent can be characterized by an expected, discounted, intertemporal utility

$$E\left[\sum_{t=0}^{T} e^{-\rho\, t}\, \frac{c_t^{1-\gamma} - 1}{1-\gamma}\right] \tag{2.25}$$

over all consumption streams $(c_0,\ \mathbf{c})$. Facing his endowment stream, an agents uses financial markets to achieve the *optimal consumption stream* $(c_0,\ \mathbf{c})$ maximizing the above utility of all achievable consumption streams in the budget set $B(w_0,\ \mathbf{w})$.

Since, by assumption, Ω is finite, Inada conditions and strict concavity guarantee existence and uniqueness of the optimal consumption stream for the objective function (2.25).

Lemma 2.18 *The stream* $(c_0,\ \mathbf{c})$ *satisfies the standard Euler equation (see, (B.20) in [30])*

$$c_t^{-\gamma}\; p_{i\,t} = E\left[e^{-\rho}\; c_{t+1}^{-\gamma}\left(p_{i\,t+1} + d_{i\,t+1}\right)\Big|\,\mathcal{G}_t\right] \tag{2.26}$$

for any asset A_i, $i = 1, \ldots, N$. *That is,* $R_t = e^{-\rho t} c_t^{-\gamma}$ *is a state price density process for the market* $(\mathcal{M}, \mathcal{G})$.

By definition, (2.24) is necessary for the utility maximization problem to be well defined. Standard results imply that it is also sufficient for the existence of the solution. In fact, the optimal consumption stream exists and satisfies the first order conditions under fairly general assumptions. See, Kramkov and Schachermayer [26], Karatzas and Zitkovic [25]. Existence proof for general probability spaces is rather complicated, but in the finite dimensional setting of our model, it is a consequence of standard convex optimization.

Using Lemma 2.13, it is possible to show that, for the class $\mathfrak{C}$, the Euler equations take a special form, indicated below (see, Malamud and Trubowitz [29, Proposition 5.2]). Furthermore, a direct calculation shows that

$$X_t(\mathbf{x}) = \mathrm{P}_{\mathcal{G}_t}\left[\sum_{\tau=t}^{T} D_{\mathbf{x}\,\tau}\,\frac{M_\tau}{M_t}\right] \tag{2.27}$$

for any portfolio strategy $\mathbf{x}$. This identity allows us to rewrite the budget constraints in a form, involving only the consumption and endowment. See, [29, Theorem 2.15 and Propositions 5.1–5.2].

Proposition 2.19 *The utility maximization problem has a solution if and only if* (2.24) *is satisfied. The optimal consumption stream* $(c_0,\ \mathbf{c})$ *is uniquely determined by the first order conditions*

$$e^{-\rho}\,E\left[u'(c_{t+1})\,|\,\mathcal{H}_{t+1}\right] = \frac{M_{t+1}}{M_t}\,u'(c_t) \tag{2.28}$$

and the budget constraints

$$(I - \mathrm{P}_{\mathcal{H}_t})\,\mathrm{P}_{\mathcal{G}_t}\left[\sum_{\tau=t}^{T}(c_\tau - w_\tau)\,M_\tau\right] = 0 \tag{2.29}$$

for all $t = 1, \ldots, T$ *and*

$$E\left[\sum_{\tau=0}^{T}(c_\tau - w_\tau)\,M_\tau\right] = 0. \tag{2.30}$$

Furthermore, $c_0 = c_0(w_0,\ \mathbf{w})$ *is monotone increasing in* w_0.

Equations (2.28)–(2.30) form a highly non linear and complicated system. Malamud and Trubowitz [29] introduced a recursive procedure for explicitly solving the system (2.28)–(2.30).

2.4 Utility indifference pricing

We start with the standard definition of the indifference price (see, e.g., [20]). Let

$$U^{\max}(w_0,\ \mathbf{w}) = E\left[\sum_{t=0}^{T} e^{-\rho\,t}\,\frac{c_t^{1-\gamma}}{1-\gamma}\right] \tag{2.31}$$

where $(c_0,\ \mathbf{c}) = (c_0(w_0,\ \mathbf{w}),\ \mathbf{c}(c_0,\ \mathbf{w}))$ is the optimal consumption stream, see also (4.30) below. Clearly, the value function $U^{\max}$ is strictly monotone increasing w_t for each t (see, e.g., Lemma 2.6 in [30]).

Consider now a company (private bank) with an initial capital (endowment) $w_0 = W$ and no random endowment $\mathbf{w} = 0$, that invests the capital into financial assets and trades in the market to achieve the optimal consumption (dividend) stream $(c_0(W, 0),\ \mathbf{c}(c_0,\ 0))$. Suppose now that this company decides to sell insurance against a $\mathcal{G}$–adapted stream $\mathbf{Y} = (Y_t,\ t = 1, \ldots, T)$ of claims for an initial, nonrandom price $\pi_0 = \pi_0(\mathbf{Y})$ at time zero. Then, the endowment stream of the company becomes

$$(w_0,\ \mathbf{w}) = (W + \pi_0,\ -\mathbf{Y})$$

and the company will trade in the market to achieve the maximal utility $U^{\max}(W + \pi_0,\ -\mathbf{Y})$.

Definition 2.20 The utility indifference price at time zero of the stream $\mathbf{Y}$ at the capital level W is the unique, deterministic solution $\pi_0 = \pi_0(W, \mathbf{Y})$ to the equation

$$U^{\max}(W, 0) = U^{\max}(W + \pi_0(W, \mathbf{Y}), -\mathbf{Y}), \tag{2.32}$$

provided it exists.

Interestingly enough, the indifference price does not always exist when $\gamma < 1$. In [30], we prove the following

Proposition 2.21 *There exists a continuous function $l = l(\mathbf{Y}, \gamma)$ such that (2.32) has a solution π_0 if and only if $W > l$. The lower threshold $l(\mathbf{Y}, \gamma) = 0$ is equal to zero if and only if either $\gamma \geq 1$ or $\mathbf{Y}$ can be replicated by trading.*

3 Main results

It is not difficult to see that the following is true

Proposition 3.1 *The premium $\pi_0(W, \mathbf{Y})$ is homogeneous of degree one. That is,*

$$\pi_0(\lambda\, W, \lambda\, \mathbf{Y}) = \lambda\, \pi_0(W, \mathbf{Y}). \tag{3.1}$$

Furthermore, it is jointly convex in capital and claims and is monotone decreasing in capital.

The main goal of this paper is to study the behavior of the premium when the claims size is large relative to the capital of the insurance company.

Note that, by Proposition 2.21 , this analysis only makes sense when $\gamma \geq 1$. When $\gamma < 1$ and the wealth W is below the threshold $l(\mathbf{Y})$, utility indifference premium simply does not exist. Therefore, everywhere in the sequel we assume that $\gamma \geq 1$.

We multiply the claims $\mathbf{Y}$ by a parameter $\lambda > 0$ measuring the size of the claims and see what happens when $\lambda \to \infty$. Homogeneity of the premium implies that we can study its behavior as the capital changes instead of analyzing the behavior when the claims size changes (see, Proposition 3.1):

$$\pi_0(W, \lambda\, \mathbf{Y}) = \lambda\, \pi_0(\lambda^{-1}\, W, \mathbf{Y}). \tag{3.2}$$

Thus, we want to study the behavior of π_0 as W goes to zero.

For the sake of brevity, we denote $c_t(W + \pi_0(W, \mathbf{Y}), -\mathbf{Y})$ by $c_t(W)$. Similarly, we denote by $X_t(W)$ the corresponding wealth process. By Proposition 2.10,

$$c_t(W) = X_t(W) - Y_t - E\left[\frac{M_{t+1}}{M_t}\, X_{t+1}(W)\,\Big|\, \mathcal{G}_t\right] \tag{3.3}$$

for $t \geq 1$ and

$$c_0(W) = W + \pi_0(W, \mathbf{Y}) - E[M_1\, X_1(W)].$$

Recall Proposition 2.16 and let $(\mathbf{Y}_t^u)$ be the minimal, upper hedging process for the claims payment stream $\mathbf{Y}$. We start with a lemma that gives the indifference premium for initial capital $W \to 0$.

Lemma 3.2 *We have*

$$\lim_{W\to 0} \pi_0(W, \mathbf{Y}) = \pi_0(0, \mathbf{Y}) = \mathbf{Y}_0^u \tag{3.4}$$

and

$$\lim_{W\to 0} X_t(W) = X_t(0) = \mathbf{Y}_t^u \tag{3.5}$$

for all $t = 1, \ldots, T$. *Consequently,*

$$\lim_{W\to 0} c_t(W) = c_t(0) = \mathbf{Y}_t^u - Y_t - E[M_{t+1}\,\mathbf{Y}_{t+1}^u\,M_t^{-1}\,|\,\mathcal{G}_t] \tag{3.6}$$

for all $t = 0, \ldots, T$. *In particular,*

$$\lim_{W\to 0} c_0(W) = c_0(0) = \mathbf{Y}_0^u - E[M_1\,\mathbf{Y}_1^u] = 0. \tag{3.7}$$

Proof Recall that the endowment stream of the company after selling the insurance against $\mathbf{Y}$ is given by

$$w_0 = W + \pi_0\ ,\ \mathbf{w} = -\mathbf{Y}.$$

By Lemma 2.17, the budget set is non-empty if and only if

$$\pi_0 + W \geq \mathbf{Y}_0^u. \tag{3.8}$$

Furthermore, $\pi_0 \leq \mathbf{Y}_0^u$, because, otherwise (2.32) does not hold. Thus, (3.4) follows. By Proposition 2.16,

$$X_t(0) \geq \mathbf{Y}_t^u \tag{3.9}$$

and, since $\pi_0(0,\ \mathbf{Y}) = \mathbf{Y}_0^u$ is sufficient to finance (X_t), Proposition 2.16 implies that $X_t(0) = \mathbf{Y}_t^u$ for all $t = 1, \ldots, T$. □

The main results of these paper are summarized in the next two theorems.

Theorem 3.3 *We have*

$$\pi - \mathbf{Y}_0^u = \begin{cases} -W + B_1(\mathbf{Y})\,W^{\alpha(\mathbf{Y})} + o(W^{\alpha(\mathbf{Y})}), & \gamma = 1 \\ (-1 + A(\mathbf{Y}))\,W + B_2(\mathbf{Y})\,W^{\gamma} + O(W^2), & \gamma \in (1,2) \\ (-1 + A(\mathbf{Y}))\,W + B_3(\mathbf{Y})\,W^2 + O(W^3), & \gamma = 2 \\ (-1 + A(\mathbf{Y}))\,W + B_4(\mathbf{Y})\,W^2 + O(W^{\min\{3,\gamma\}}), & \gamma > 2. \end{cases} \tag{3.10}$$

Here, $\alpha(\mathbf{Y})$ *is given by* (7.16) *below,*

$$B_1(\mathbf{Y}) = \alpha^{-1}\,c_0^{(\alpha)}\,(1 + \langle \mathfrak{d},\ \mathbf{1}\rangle) \tag{3.11}$$

and

$$A(\mathbf{Y}) = \left(c_0^{(1)}\right)^{\gamma}\,Z_0^{\gamma} \tag{3.12}$$

and

$$B_2(\mathbf{Y}) = c_0^{(\gamma)}\,\left(c_0^{(1)}\right)^{\gamma-1}\,Z_0^{\gamma} \tag{3.13}$$

and

$$B_4(\mathbf{Y}) = \frac{\gamma}{2}c_0^{(2)}\,\left(c_0^{(1)}\right)^{\gamma-1}\,Z_0^{\gamma}. \tag{3.14}$$

The coefficients $c_0^{(1)},\ c_0^{(\alpha)},\ c_0^{(\gamma)}$ *are constructed in the Appendix C via an explicit, recursive procedure. The coefficient* $B_3(\mathbf{Y})$ *also satisfies* (3.14)*, but the coefficient* $c_0^{(2)}$ *is a little bit different from the one, calculated in Lemma* 7.7.

The next theorem shows that, for $\gamma \in (1, 2)$, the asymptotic of Theorem 3.3 provides sharp, non-perturbative bounds for the indifference price.

Theorem 3.4 *Let $\gamma > 1$ and $A(\mathbf{Y}) = \left(c_0^{(1)}(\mathbf{Y})\, Z_0\right)^{\gamma}$. Then,*

$$\begin{aligned} &\mathbf{Y}_0^u - W\,(1 - A(\mathbf{Y})) \le \pi_0(W, \mathbf{Y}) \\ &\le \mathbf{Y}_0^u - W\left(1 - A(\mathbf{Y})\left(1 - (\gamma - 1)\left(c_0^{(\gamma)}/c_0^{(1)}\right) W^{\gamma-1}\right)^{\frac{1}{1-\gamma}}\right). \end{aligned}$$

To prove the above results, we will need to get sharp analytical control of the optimal consumption stream. This is done in the subsequent sections.

4 Separating consumption at time zero

Consumption at time zero plays a very special role in the structure of the optimal consumption stream. We illustrate this on the simple example of a complete market.

If the market is complete, $\mathcal{H}_t = \mathcal{G}_t$, and the first order conditions (2.28) take the simple form

$$e^{-\rho t}\, c_t^{-\gamma} = M_t\, c_0^{-\gamma} \;\Leftrightarrow c_t = e^{-\rho t/\gamma}\, M_t^{-1/\gamma}\, c_0. \tag{4.1}$$

Furthermore, the single, intertemporal budget constraint (2.30) implies that

$$c_0 = \frac{w_0 + E\left[\sum_{t=1}^T w_t\, M_t\right]}{1 + E\left[\sum_{t=1}^T e^{-\rho t/\gamma}\, M_t^{1-1/\gamma}\right]}. \tag{4.2}$$

Thus, endowment process $(w_0,\ \mathbf{w})$ only enters the optimal consumption stream through c_0, and c_0 is a linear function of the endowment stream. Thus, it is natural to write the optimal consumption stream $\mathbf{c} = (c_t,\ t = 1, \ldots, T)$ in the form

$$\mathbf{c} = \mathbf{c}(c_0) \qquad \text{and} \qquad c_0 = c_0(w_0,\ \mathbf{w}). \tag{4.3}$$

It turns out that a similar representation is possible in general incomplete markets. This representation plays a crucial role in our analysis.

4.1 Notations and definitions

Let

$$H = \oplus_{t=1}^T L_2(\Omega, \mathcal{G}_t) \tag{4.4}$$

be the Hilbert space of all adapted processes, starting at $t = 1$, equipped with the standard inner product

$$\langle \mathbf{Z}, \mathbf{Y} \rangle = \sum_{t=1}^T E[Z_t Y_t] = \sum_{t=1}^T \langle Z_t, Y_t \rangle, \tag{4.5}$$

for any $\mathbf{Z} = (Z_t),\ \mathbf{Y} = (Y_t) \in H$. Any $\mathcal{G}$-adapted process

$$\mathbf{a} = (a_1, \ldots, a_T) \tag{4.6}$$

defines a natural multiplication operator on H via

$$\mathbf{a}\,\mathbf{Z} = \text{diag}(a_t)_{t=1}^T\,\mathbf{Z} = (a_t\,Z_t) \in H. \tag{4.7}$$

We will also use the operator

$$\mathfrak{d} = \text{diag}(e^{-\rho t})_{t=1}^T. \tag{4.8}$$

Depending on the context, we use boldface letters to denote both vectors and the corresponding multiplication operators.

The following special inner product plays a crucial role in our analysis.

Definition 4.1 Fix an endowment stream $(w_0,\ \mathbf{w})$. Let

$$(c_0,\ \mathbf{c}) = (c_0(w_0,\ \mathbf{w}),\ \mathbf{c}(c_0,\ \mathbf{w})) \tag{4.9}$$

be the corresponding optimal consumption stream, defined in Proposition 2.19. We define the inner product

$$\langle \mathbf{Z}, \mathbf{Y}\rangle_c = \sum_{t=1}^T e^{-\rho t}\, E\left[c_t^{-\gamma-1}\, Z_t\, Y_t\right] = \left\langle \mathfrak{d}\mathbf{c}^{-\gamma-1}\,\mathbf{Z},\ \mathbf{Y}\right\rangle. \tag{4.10}$$

Remark 4.2 We emphasize that the inner product depends on the endowment stream. Thus, it should be viewed as a Riemannian structure on the space of all endowment streams: in each point of the space, there is a metric, defined by the inner product (4.10).

The following lemma in an immediate consequence of the definition.

Lemma 4.3 *Let $\gamma \neq 1$. Then, the norm squared of the optimal consumption stream $\mathbf{c}$ is given by*

$$\langle \mathbf{c}, \mathbf{c}\rangle_c = \sum_{t=1}^T e^{-\rho t} E\left[c_t^{1-\gamma}\right] = (1-\gamma)U(c_0,\ \mathbf{c}) - c_0^{1-\gamma}, \tag{4.11}$$

where

$$U(c_0,\ \mathbf{c}) = (1-\gamma)^{-1} \sum_{t=1}^T e^{-\rho t}\, E\left[c_t^{1-\gamma}\right] \tag{4.12}$$

is the maximal utility, achievable by trading, of an agent with endowment $(w_0,\ \mathbf{w})$.

In the sense of Lemma 4.3, the inner product $\langle\cdot,\cdot\rangle_c$ is an economically natural inner product: the size (norm) of the consumption stream is equal to its utility.

Let $J\ :\ H\ \to\ H$ be the linear operator defined by

$$(J\mathbf{Z})_t = \sum_{\tau=1}^t Z_\tau \tag{4.13}$$

for $t = 1, \ldots, T$. It is easy to see that the adjoint operator J^* of J with respect to the standard inner product is given by

$$\left(J^*\mathbf{Z}\right)_t = \mathrm{P}_{\mathcal{G}_t} \sum_{\tau=t}^T Z_\tau \quad \text{with} \quad \langle J\,\mathbf{Z},\ \mathbf{Y}\rangle = \left\langle \mathbf{Z},\ J^*\mathbf{Y}\right\rangle. \tag{4.14}$$

Let for $t = 1, \ldots, T$,

$$Q_t = \mathrm{P}_{\mathcal{G}_t} - \mathrm{P}_{\mathcal{H}_t} \tag{4.15}$$

and let $\mathbf{Q} : H \to H$ be the orthogonal sum

$$\mathbf{Q} = \oplus_{t=1}^{T} Q_t. \tag{4.16}$$

The image

$$H_0 = \mathbf{Q}H = \oplus_{t=1}^{T} Q_t\, L_2(\Omega, \mathcal{G}_t) \tag{4.17}$$

of the orthogonal projection $\mathbf{Q}$ will play an important role in our analysis. Intuitively, this is the "unhedgeable" subspace.

Let H_1, H_2 be two Hilbert spaces. Given a smooth map $G : H_1 \to H_2$, we will use

$$D(G) = \frac{\partial G(\mathbf{w})}{\partial \mathbf{w}} : H_1 \to H_2 \tag{4.18}$$

and

$$D^2(G) = \frac{\partial^2 G(\mathbf{w})}{\partial \mathbf{w}^2} : H_1 \times H_1 \to H_2 \tag{4.19}$$

to denote its first and second Fréchet derivatives.

4.2 Construction of optimal consumption streams

The goal of this section is to understand the structure of the nonlinear map

$$(w_0, \mathbf{w}) \to (c_0, \mathbf{c}) \tag{4.20}$$

mapping the endowment stream into the optimal consumption stream, defined in Proposition 2.19. This is analogous to the complete market case (see, (4.1)). The recursive construction of [29] explains its local structure, i.e., the dependence between c_t and c_{t+1}. In this section we introduce a new formalism that allows to treat this map in a global way and derive interesting properties of its derivatives, that can not be seen in the "local", recursive formalism of [29].

As we explain above, one of the key ideas is to decouple the initial consumption and construct the map $\mathbf{c} = \mathbf{c}(c_0, \mathbf{w})$.

We start with a simple

Lemma 4.4 *There exists a function $A : H \to \mathbb{R}$ such that for every $x \in (A(\mathbf{w}), +\infty)$ there exists a unique number (see, (2.24))*

$$w_0 > (-\mathbf{w})_0^u \tag{4.21}$$

for which the optimal consumption stream $(c_0, \mathbf{c})$, corresponding to the endowment process $(w_0, \mathbf{w})$, has initial consumption $c_0 = x$.

Fix a consumption c_0 at time zero and let

$$\mathbf{c_M} = \mathbf{c_M}(c_0) = c_0\, \mathfrak{d}^{1/\gamma} \mathbf{M}^{-1/\gamma} \tag{4.22}$$

be the optimal consumption stream in a *fictitious* complete market with the unique SPD process $\mathbf{M}$ (see, (4.1)).

Definition 4.5 Let $H_0^+ = \{\mathbf{Z} \in H_0;\ \mathbf{1} + J\mathbf{Z} > 0\}$. Define the map $F : H_0^+ \to H_0$ via

$$F(\mathbf{Z}) = F_{c_0}(\mathbf{Z}) = \mathbf{Q}J^*\, \mathbf{M}\, \mathbf{c_M}\, (\mathbf{1} + J\mathbf{Z})^{-1/\gamma}. \tag{4.23}$$

A direct calculation shows that the following is true.

Lemma 4.6 *The map $F : H_0^+ \to F(H_0^+) \subset H_0$ is bijective and monotone decreasing, in the sense that for all $\mathbf{Z}, \mathbf{Y} \in H_0^+$*

$$\langle F(\mathbf{Z}) - F(\mathbf{Y}),\ \mathbf{Z} - \mathbf{Y}\rangle \leq 0. \tag{4.24}$$

The inequality is strict as soon as $\mathbf{Z} \neq \mathbf{Y}$.

Proposition 4.7 *Let $\mathbf{w} = (w_t,\ t = 1, \ldots, T)$ be an endowment process. Choose $c_0 > A(\mathbf{w})$ and let w_0 be the corresponding initial endowment. Then,*

$$\mathbf{Q}J^*\mathbf{M}\mathbf{w}\ \in\ F_{c_0}(H_0^+) \tag{4.25}$$

and the optimal consumption stream $\mathbf{c}$ is given by

$$\mathbf{c} = \mathbf{c}(c_0,\ \mathbf{w}) =\ \mathbf{c}_{\mathbf{M}}(c_0)\,(1 + J\,\mathbf{Z}(c_0,\ \mathbf{w}))^{-1/\gamma} \tag{4.26}$$

with

$$\mathbf{Z}(\mathbf{w}) = \mathbf{Z}(c_0,\ \mathbf{w}) = F_{c_0}^{-1}\left(\mathbf{Q}J^*\mathbf{M}\mathbf{w}\right) = F^{-1}\left(\mathbf{Q}J^*\mathbf{M}\mathbf{w}\right)\ \in H_0^+. \tag{4.27}$$

The derivatives of the maps $\mathbf{c}$ and F are given by

$$D(\mathbf{c}) = \frac{\partial \mathbf{c}(c_0,\ \mathbf{w})}{\partial \mathbf{w}} = -\gamma^{-1}\mathbf{c}_{\mathbf{M}}{}^{-\gamma}\mathbf{c}^{1+\gamma}\ J\,(D(F))^{-1}\,\mathbf{Q}\,J^*\,\mathbf{M} \tag{4.28}$$

and

$$D(F) = \frac{\partial F}{\partial \mathbf{Z}} = -\gamma^{-1}\mathbf{Q}\,J^*\,\mathbf{M}\ \mathbf{c}_{\mathbf{M}}{}^{-\gamma}\ \mathbf{c}^{1+\gamma}\ J \tag{4.29}$$

respectively.

Remark 4.8 Proposition 4.7 implies that the optimal consumption stream can be written in the form

$$(c_0, \mathbf{c}) = (c_0(w_0, \mathbf{w}),\ \mathbf{c}(c_0(w_0, \mathbf{w}),\ \mathbf{w})). \tag{4.30}$$

Given $\mathbf{c} = \mathbf{c}(c_0,\ \mathbf{w})$, the value $c_0 = c_0(w_0,\ \mathbf{w})$ is uniquely determined by the last budget constraint (2.30). This is similar to the complete market situation (see, (4.1) and (4.2)).

We are now ready to state the main result of this section.

Theorem 4.9 *Let $c_0 > A(\mathbf{w})$ and $(c_0, \mathbf{w}) \mapsto\ \mathbf{c}(c_0,\ \mathbf{w})$ be the map defined in Proposition 4.7. Then, the derivative $D(\mathbf{c}) = \partial\mathbf{c}/\partial\mathbf{w}$, given in (4.28), is the orthogonal projection $\mathbf{P_c}$ onto the subspace*

$$H_c = \mathbf{M}\left(\partial\mathbf{c}^{-\gamma-1}\right)^{-1} J\,H_0 \tag{4.31}$$

in the Hilbert space $(H, \langle\cdot, \cdot\rangle_c)$, equipped with the inner product (4.10).

4.3 The optimal consumption stream without a random endowment

To calculate the solution to (2.32), we need to know the exact value of the left hand side. It is well known that, for a diffusion driven incomplete market (see, the discussion after Example 2.9) without a random endowment, the optimal consumption stream for a logarithmic utility ($\gamma = 1$) can be calculated explicitly. For $\gamma \neq 1$, no explicit expression is known.

The special structure of incomplete markets in the class $\mathfrak{C}$ allows us to explicitly solve the utility maximization problem for a CRRA utility without a random endowment.

Proposition 4.10 *Let* $(c_0, \mathbf{c})$ *be the optimal consumption stream for the endowment* $(W, 0)$, *maximizing* (2.31) *and* X_t *be the corresponding wealth process, i.e.,*

$$c_t = X_t - E\left[\frac{M_{t+1}}{M_t} X_{t+1} \,\Big|\, \mathcal{G}_t\right]. \tag{4.32}$$

Let $Z_T = 1$. *Define the process* $\mathbf{Z} = (Z_t,\ t = T-1, \ldots, 0)$ *inductively by*

$$Z_t = 1 + e^{-\rho/\gamma^{-1}} E\left[\left(\frac{M_{t+1}}{M_t}\right)^{1-1/\gamma} \left(E\left[Z_{t+1}^{\gamma} \,|\, \mathcal{H}_{t+1}\right]\right)^{1/\gamma} \,\Big|\, \mathcal{G}_t\right]. \tag{4.33}$$

Then,

$$X_t = X_{t-1}\, e^{-\rho/\gamma} \left(\frac{M_t}{M_{t-1}}\right)^{-1/\gamma} \left(E\left[Z_t^{\gamma} \,|\, \mathcal{H}_t\right]\right)^{1/\gamma} Z_{t-1}^{-1} \tag{4.34}$$

(with $X_0 = W$*) and*

$$c_t = c_{t-1}\, e^{-\rho/\gamma} \left(\frac{M_t}{M_{t-1}}\right)^{-1/\gamma} \left(\frac{E\left[Z_t^{\gamma} \,|\, \mathcal{H}_t\right]}{Z_t^{\gamma}}\right)^{1/\gamma} \tag{4.35}$$

for all $t \geq 1$ *and*

$$c_0 = W\, Z_0^{-1}. \tag{4.36}$$

In particular, for $\gamma = 1$,

$$c_t = e^{-\rho\, t}\, M_t^{-1}\, c_0 \tag{4.37}$$

and

$$c_0 = \frac{W}{\sum_{t=0}^{T} e^{-\rho\, t}}. \tag{4.38}$$

Proof Since the aggregate state price density process $\mathbf{M}$ is $\mathcal{H}$-adapted, (4.34) implies that (X_t) is also $\mathcal{H}$-adapted. Identities (4.34) and (4.35) imply

$$\frac{c_t}{X_t} = \frac{c_{t-1}}{X_{t-1}} \frac{Z_{t-1}}{Z_t}$$

and, consequently,

$$c_t = X_t\, Z_t^{-1}.$$

Using (4.33), we get

$$
\begin{aligned}
&X_{t-1} - E\left[\frac{M_t}{M_{t-1}}\, X_t \,\middle|\, \mathcal{G}_{t-1}\right] \\
&= X_{t-1}\left(1 - E\left[\left(\frac{M_t}{M_{t-1}}\right)^{1-1/\gamma} e^{-\rho/\gamma}\, \left(E\left[Z_t^{\gamma}\,|\,\mathcal{H}_t\right]\right)^{1/\gamma}\, Z_{t-1}^{-1} \,\middle|\, \mathcal{G}_{t-1}\right]\right) \\
&= \frac{X_{t-1}}{Z_{t-1}}\left(Z_{t-1} - E\left[\left(\frac{M_t}{M_{t-1}}\right)^{1-1/\gamma} e^{-\rho/\gamma}\, \left(E\left[Z_t^{\gamma}\,|\,\mathcal{H}_t\right]\right)^{1/\gamma} \,\middle|\, \mathcal{G}_{t-1}\right]\right) \\
&= \frac{X_{t-1}}{Z_{t-1}} = c_{t-1},
\end{aligned}
$$

and thus, (c_t) is indeed the consumption stream, corresponding to the wealth process (X_t). It follows directly from (4.35) that (c_t) satisfies the first order conditions and the claim follows. □

Corollary 4.11 *If* $\gamma \neq 1$ *then*

$$U^{\max}(W,\,0) = (1-\gamma)^{-1}\, W^{1-\gamma}\, Z_0^{\gamma}. \tag{4.39}$$

If $\gamma = 1$, *then*

$$U^{\max}(W,\,0) = \sum_{t=0}^{T} e^{-\rho\, t}\, \log\, W + \sum_{t=0}^{T} e^{-\rho\, t}\, \log\left(\frac{e^{-\rho\, t}\, M_t}{\sum_{s=0}^{T} e^{-\rho\, s}}\right). \tag{4.40}$$

Proof By (5.12),

$$(c_0 - W)\, c_0^{-\gamma} + \langle\, \mathbf{c},\, \mathfrak{d}\, \mathbf{c}^{-\gamma} \rangle = 0. \tag{4.41}$$

Substituting (4.36), we get

$$(1-\gamma)\, U^{\max}(W,\,0) = W\, c_0^{-\gamma} = W^{1-\gamma}\, Z_0^{\gamma}. \tag{4.42}$$

The case $\gamma = 1$ is proved by direct calculation. □

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Appendix A: Proof of Theorem 4.9

We need an auxiliary

Lemma 5.1 *The adjoint of* $\mathbf{P_c} = D(\mathbf{C})|_{\mathbf{w}}$ *with respect to the standard inner product* $\langle\,\cdot,\,\cdot\rangle$ *is given by*

$$\mathbf{P_c^*} = -\gamma^{-1}\, \mathbf{M}\, J\, D(F)^{-1}\, \mathbf{Q}\, J^*\, \mathbf{c}^{\gamma+1}\, \mathbf{c_M}^{-\gamma}. \tag{5.1}$$

Moreover,

$$\mathbf{M}\, \mathbf{c_M}^{\gamma}\, \mathbf{c}^{-\gamma-1}\, \mathbf{P_c} = \mathbf{P_c^*}\, \mathbf{M}\, \mathbf{c_M}^{\gamma}\, \mathbf{c}^{-\gamma-1}. \tag{5.2}$$

Proof of Theorem 4.9 The operator $\mathbf{Q}\,J^*\,\mathbf{M}$ maps H onto H_0 and therefore, $J\,(D(F))^{-1}\,\mathbf{Q}\,J^*\,\mathbf{M}$ maps H onto $J\,H_0$. Substituting the identity $\mathbf{c_M}^{-\gamma}\mathbf{c}^{1+\gamma} = c_0^{-\gamma}\,\mathfrak{d}^{-1}\mathbf{M}\mathbf{c}^{1+\gamma}$ into (4.28), we immediately get that the $D(\mathbf{c})$ maps H onto H_c.

It remains to prove that $D(\mathbf{c})$ is an orthogonal projection. Identity (4.29) implies that

$$\mathbf{Q} = D(F)\,(D(F))^{-1}\,\mathbf{Q} = -\gamma^{-1}\mathbf{Q}J^*\mathbf{M}\,\mathbf{c_M}^{-\gamma}\mathbf{c}^{1+\gamma}J\;D(F)^{-1}\mathbf{Q}. \tag{5.3}$$

Multiplying (5.3) from the left and right with $-\gamma^{-1}\mathbf{c_M}^{-\gamma}\mathbf{c}^{1+\gamma}J\;\;D(F)^{-1}$ and $J^*\,\mathbf{M}$ respectively, we obtain

$$\begin{aligned}&-\gamma^{-1}\mathbf{c_M}^{-\gamma}\mathbf{c}^{1+\gamma}J\;D(F)^{-1}\mathbf{Q}J^*\mathbf{M}\\ &= \gamma^{-2}\mathbf{c_M}^{-\gamma}\mathbf{c}^{1+\gamma}J\;D(F)^{-1}\mathbf{Q}J^*\mathbf{M}\mathbf{c_M}^{-\gamma}\mathbf{c}^{1+\gamma}J\;D(F)^{-1}\mathbf{Q}J^*\mathbf{M}.\end{aligned} \tag{5.4}$$

That is, $D(\mathbf{c}) = D(\mathbf{c})^2$.

It remains to prove that $D(\mathbf{c})$ is self-adjoint w.r.t. $\langle\cdot,\cdot\rangle_c$. By Lemma 5.1,

$$\mathbf{M}\,\mathbf{c_M}^{\gamma}\;\mathbf{c}^{-\gamma-1}D(\mathbf{c}) = D^*(\mathbf{c})\mathbf{M}\,\mathbf{c_M}^{\gamma}\;\mathbf{c}^{-\gamma-1}, \tag{5.5}$$

where $D^*(\mathbf{c})$ is the adjoint with respect to the standard inner product. Therefore, using (4.22), we get

$$\begin{aligned}\langle D(\mathbf{c})\mathbf{X},\mathbf{Y}\rangle_c &= \left\langle\mathfrak{d}\mathbf{c}^{-\gamma-1}D(\mathbf{c})\mathbf{X},\mathbf{Y}\right\rangle = \left\langle\mathbf{M}\,\mathbf{c_M}^{\gamma}\;\mathbf{c}^{-\gamma-1}D(\mathbf{c})\mathbf{X},\mathbf{Y}\right\rangle\\ &= \left\langle\mathbf{M}\,\mathbf{c_M}^{\gamma}\;\mathbf{c}^{-\gamma-1}\mathbf{X},D(\mathbf{c})\mathbf{Y}\right\rangle = \left\langle\mathfrak{d}\mathbf{c}^{-\gamma-1}\mathbf{X},D(\mathbf{c})\mathbf{Y}\right\rangle = \langle\mathbf{X},D(\mathbf{c})\mathbf{Y}\rangle_c\end{aligned} \tag{5.6}$$

which is what had to be proved. □

Differentiating the representation of $\mathbf{c}$ from Proposition 4.7 and using the fact that $\mathbf{P_c}$ is a projection, it is possible to show that the following is true.

Lemma 5.2 *Under the assumptions of Proposition 4.7, the second derivative of* $\mathbf{c}(c_0,\,\mathbf{w})$ *with respect to* $\mathbf{w}$ *is given by*

$$\frac{\partial^2\mathbf{c}(c_0,\,\mathbf{w})}{\partial\mathbf{w}^2}(\mathbf{Y},\mathbf{Y}) = (1+\gamma)\,(I-\mathbf{P_c})\,\mathbf{c}^{-1}\,(\mathbf{P_c}\mathbf{Y})^2\,. \tag{5.7}$$

The following result follows directly from homogeneity of a CRRA utility.

Lemma 5.3 *Let* $\mathbf{w}$ *be an endowment stream and* $c_0 > A(\mathbf{w})$. *The map* $\mathbf{c}(c_0,\,\mathbf{w})$ *is homogeneous of degree one, that is,*

$$\mathbf{c}(c_0,\,\mathbf{w}) = c_0\,\mathbf{c}(1,\,c_0^{-1}\,\mathbf{w}). \tag{5.8}$$

Consequently, the Euler identity

$$\frac{\partial\mathbf{c}(c_0,\,\mathbf{w})}{\partial c_0} = c_0^{-1}\,\mathbf{c}(c_0,\,\mathbf{w})\;-\;c_0^{-1}\,\frac{\partial\mathbf{c}(c_0,\,\mathbf{w})}{\partial\mathbf{w}}(\mathbf{w}) \tag{5.9}$$

holds, as well as

$$\frac{\partial^2\mathbf{c}(c_0,\,\mathbf{w})}{\partial c_0\,\partial\mathbf{w}}(\mathbf{y}) = -c_0^{-1}\,\frac{\partial^2\mathbf{c}(c_0,\,\mathbf{w})}{\partial\mathbf{w}^2}(\mathbf{w},\,\mathbf{y}), \tag{5.10}$$

and

$$\frac{\partial^2\mathbf{c}(c_0,\,\mathbf{w})}{\partial c_0^2} = c_0^{-2}\,\frac{\partial^2\mathbf{c}(c_0,\,\mathbf{w})}{\partial\mathbf{w}^2}(\mathbf{w},\,\mathbf{w}). \tag{5.11}$$

By Corollary 2.18, $\mathbf{R} = (c_0^{-\gamma}, \mathfrak{d}\, \mathbf{c}^{-\gamma})$ is an SPD process and we arrive at:

Lemma 5.4 *The optimal consumption stream satisfies*

$$(w_0 - c_0)\, c_0^{-\gamma} + \left\langle \mathbf{w} - \mathbf{c}(c_0,\, \mathbf{w}),\, \mathfrak{d}\, \mathbf{c}(c_0,\, \mathbf{w})^{-\gamma} \right\rangle \;=\; 0. \tag{5.12}$$

We also need the following identity.

Lemma 5.5 *Let* $\mathbf{c} = \mathbf{c}(c_0,\, \mathbf{w})$. *Then,*

$$\mathbf{P_c}\, \mathbf{w} = \mathbf{P_c}\, \mathbf{c}. \tag{5.13}$$

Proof The budget constraints (2.29) can be rewritten as

$$\mathbf{Q}\, J^*\, \mathbf{M}\, \mathbf{c} = \mathbf{Q}\, J^*\, \mathbf{M}. \tag{5.14}$$

Using (4.28), we arrive at

$$\begin{aligned} \mathbf{P_c w} &= -\gamma^{-1} \mathbf{c_M}^{-\gamma} \mathbf{c}^{1+\gamma}\, J\, (D(F))^{-1}\, \mathbf{Q}\, J^*\, \mathbf{Mw} \\ &= -\gamma^{-1} \mathbf{c_M}^{-\gamma} \mathbf{c}^{1+\gamma}\, J\, (D(F))^{-1}\, \mathbf{Q}\, J^*\, \mathbf{Mc} = \mathbf{P_c c} \end{aligned} \tag{5.15}$$

which is what had to be proved. □

Definition 5.6 Let

$$\mathbf{b} = (I - \mathbf{P_c})\, (\mathbf{c}). \tag{5.16}$$

The random process **b** plays a very important role in our analysis and appears in almost every formula. We will need a

Lemma 5.7 *The process* **b** *is nonnegative.*

Proof By (5.9),

$$\frac{\partial \mathbf{c}(c_0,\, \mathbf{w})}{\partial c_0} = c_0^{-1}\, \mathbf{b}. \tag{5.17}$$

By [29, Theorem 5.14], $\mathbf{c}$ is a coordinate-wise monotone increasing function of c_0 and the claim immediately follows. □

Appendix B: Proof of Theorem 3.4

We will need the following result, which is also of independent interest.

Proposition 6.1 *Let* $\gamma > 1$. *Then, the quotient* $W^{-1}\, c_0(W + \pi_0(W,\, \mathbf{Y}),\, \mathbf{Y})$ *is monotone increasing in* W *and*

$$Z_0^{-1} = \lim_{W \to +\infty} \frac{c_0(W)}{W} \geq \frac{c_0(W)}{W} \geq \lim_{W \to 0} \frac{c_0(W)}{W} = c_0^{(1)}(\mathbf{Y}), \tag{6.1}$$

where Z_0 *is defined by* (4.33) *and* (4.36).

Proof By Proposition 3.1, π_0 is convex in W. Therefore, $\partial\pi_0/\partial W$ is monotone increasing in W and, by (7.45), so is $c_0(W)/W$. The limit on the right hand side of (6.1) follows from Lemma 7.4.

Using homogeneity, we get

$$\begin{aligned} c_0(W)/W &= c_0(W + \pi(W,\ \mathbf{Y}), -\mathbf{Y})\, W^{-1} \\ &= c_0\big(W^{-1}\,(W + \pi(W,\ \mathbf{Y})), -\mathbf{Y}\, W^{-1}\big) \\ &= c_0\big((1 + \pi(1,\ \mathbf{Y}\, W^{-1})), -\mathbf{Y}\, W^{-1}\big). \end{aligned}$$

Therefore, by (4.36),

$$\lim_{W\to+\infty} \frac{c_0(W)}{W} = c_0(1,\ 0) = Z_0^{-1}. \tag{6.2}$$

□

Combining Lemma 7.8 with Proposition 6.1, we immediately get

Proposition 6.2 *We have*

$$\lim_{W\to 0} \frac{\partial\pi_0(W,\ \mathbf{Y})}{\partial W} = -1 + \left(c_0^{(1)}(\mathbf{Y})\, Z_0\right)^{\gamma}. \tag{6.3}$$

Consequently, the following inequality always holds:

$$\mathbf{Y}_0^u \ge \pi_0(W,\ \mathbf{Y}) \ge \mathbf{Y}_0^u - W\left(1\ -\ \left(c_0^{(1)}(\mathbf{Y})\, Z_0\right)^{\gamma}\right). \tag{6.4}$$

Asymptotic expansion of Theorem 3.3 is a local result that only holds when W is small. It turns out that it is possible to prove sharp, global bounds for the premium using some interesting convexity properties of the function $c_0(W)/W$. The proof of this result is based on surprising algebraic identities for the derivatives of $c_0(W)$, leading to numerous cancellations.

Proposition 6.3 *Let* $\gamma > 1$. *Fix* $\mathbf{Y}$ *and let*

$$k(W) = k(W,\ \mathbf{Y}) = c_0(W + \pi_0(W,\ \mathbf{Y}), -\mathbf{Y}). \tag{6.5}$$

Let further

$$g(v) = \left(\frac{k\big(v^{1/(\gamma-1)}\big)}{v^{1/(\gamma-1)}}\right)^{1-\gamma}. \tag{6.6}$$

Then, the function $g(v)$ *is convex and satisfies*

$$g(0) = \big(c_0^{(1)}\big)^{1-\gamma},\quad g'(0) = (1-\gamma)\,\big(c_0^{(1)}\big)^{-\gamma}\, c_0^{(\gamma)}. \tag{6.7}$$

Proof By abuse of notation, we will use c_0 to denote the value of $k(W)$ when we do not have to differentiate. By Lemma 4.3,

$$k^{1-\gamma} + \langle \mathbf{c}^{1-\gamma}(k, -\mathbf{Y}),\ \mathfrak{d}\rangle = W^{1-\gamma}\, Z_0^{\gamma} = c_0^{1-\gamma} + \|\mathbf{c}\|_c^2. \tag{6.8}$$

Identity (5.17) implies that

$$\frac{\partial \mathbf{c}(k, -\mathbf{Y})}{\partial k} = k^{-1}\,(I - \mathbf{P_c})\,\mathbf{c}. \tag{6.9}$$

Differentiating (6.8) with respect to W, we get

$$k' = \frac{k\, W^{-\gamma}\, Z_0^{\gamma}}{k^{1-\gamma} + \|\mathbf{b}\|_c^2} = c_0\, W^{-1}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1} \tag{6.10}$$

with

$$N = k^{1-\gamma} + \|\mathbf{b}\|_c^2.$$

Differentiating (6.10) with respect to W, we get

$$\begin{aligned} k'' = \;& N^{-2} \Big((W^{-\gamma}\, Z_0^{\gamma}\, k' - \gamma\, k\, W^{-\gamma-1}\, Z_0^{\gamma})\, N \\ & - k\, W^{-\gamma}\, Z_0^{\gamma}\, ((1-\gamma) k^{-\gamma}\, k' + (\|\mathbf{b}\|_c^2)') \Big) \\ = \;& c_0\, W^{-1}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-2} \Big(\Big(W^{-1}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1} - \gamma\, W^{-1} \Big)\, N \\ & - \big((1-\gamma)\, W^{-1}\, c_0^{1-\gamma}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1} + (\|\mathbf{b}\|_c^2)' \big) \Big). \end{aligned}$$

By (5.9), (5.11) and (5.5),

$$\frac{\partial \mathbf{c}}{\partial c_0} = c_0^{-1}\, \mathbf{b} \quad \text{and} \quad \frac{\partial^2 \mathbf{c}}{\partial c_0^2} = c_0^{-2}\, (1+\gamma)\, (I - \mathbf{P_c})\, \mathbf{c}^{-1}\; (\mathbf{P_c}\, \mathbf{c})^2 .$$

Therefore, using the fact that $\mathbf{P_c}$ is an orthogonal projection with respect to $\langle \cdot, \cdot \rangle_c$, we get

$$\begin{aligned} (\|\mathbf{b}\|^2)' = \;& \frac{\partial}{\partial W} \langle k\, \frac{\partial \mathbf{c}}{\partial c_0}, \mathfrak{d}\, \mathbf{c}^{-\gamma} \rangle \\ = \;& k'\, k^{-1}\, \|\mathbf{b}\|_c^2 + k\, \langle \frac{\partial^2 \mathbf{c}}{\partial c_0^2}\, k', \mathfrak{d}\, \mathbf{c}^{-\gamma} \rangle - \gamma\, \langle k\, \frac{\partial \mathbf{c}}{\partial c_0}, \mathfrak{d}\, \mathbf{c}^{-\gamma-1}\, \frac{\partial \mathbf{c}}{\partial c_0}\, k' \rangle \\ = \;& k'\, k^{-1}\, \|\mathbf{b}\|_c^2 + k'\, k^{-1}\, (1+\gamma)\, \langle \mathbf{c}^{-1}\, (\mathbf{P_c c})^2, \mathbf{b} \rangle_c - \gamma\, k'\, k^{-1}\, \|\mathbf{b}\|_c^2 \\ = \;& (1-\gamma)\, k'\, k^{-1}\, \|\mathbf{b}\|_c^2 + k'\, k^{-1}\, (1+\gamma)\, \langle \mathbf{c}^{-1}\, (\mathbf{P_c c})^2, \mathbf{b} \rangle_c \\ = \;& W^{-1}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1}\, \big((1-\gamma)\, \|\mathbf{b}\|_c^2 + (1+\gamma)\, \langle \mathbf{c}^{-1}\, (\mathbf{P_c c})^2, \mathbf{b} \rangle_c \big). \end{aligned}$$

Consequently,

$$\begin{aligned} & (1-\gamma)\, W^{-1}\, c_0^{1-\gamma}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1} + (\|\mathbf{b}\|_c^2)' \\ & = W^{-1}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1}\, \big((1-\gamma)\, N + (1+\gamma)\, \langle \mathbf{c}^{-1}\, (\mathbf{P_c c})^2, \mathbf{b} \rangle_c \big), \end{aligned} \tag{6.11}$$

and thus

$$\begin{aligned} k'' = \;& c_0\, W^{-1}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-2} \Big(\Big(W^{-1}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1} - \gamma\, W^{-1} \Big)\, N \\ & - \big(W^{-1}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1}\, \big((1-\gamma)\, N + (1+\gamma)\, \langle \mathbf{c}^{-1}\, (\mathbf{P_c c})^2, \mathbf{b} \rangle_c \big) \big) \Big) \\ = \;& c_0\, W^{-2}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-2} \Big(\gamma\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2) - \gamma\, N \\ & - (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1}\, (1+\gamma)\, \langle \mathbf{c}^{-1}\, (\mathbf{P_c c})^2, \mathbf{b} \rangle_c \Big) \\ = \;& c_0\, W^{-2}\, (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-2} \Big(\gamma\, \|\mathbf{P_c c}\|_c^2 \\ & - (c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1}\, (1+\gamma)\, \langle \mathbf{c}^{-1}\, (\mathbf{P_c c})^2, \mathbf{b} \rangle_c \Big). \end{aligned}$$

Now,

$$(k/W)' = \frac{k'\,W - k}{W^2} = W^{-2}\,c_0\,\|\mathbf{P_c}\,\mathbf{C}\|_c^2\,N^{-1} \tag{6.12}$$

and

$$\begin{aligned}
W^3\,(k/W)'' &= k''W^2 - 2\,k'W + 2\,k\\
&= \ c_0\,(c_0^{1-\gamma} + \|\mathbf{C}\|_c^2)\,N^{-2}\Big(\gamma\,\|\mathbf{P_c C}\|_c^2\\
&\quad - (c_0^{1-\gamma} + \|\mathbf{C}\|_c^2)\,N^{-1}\,(1+\gamma)\,\langle\,\mathbf{C}^{-1}\,(\mathbf{P_c C})^2,\ \mathbf{b}\rangle_c\Big)\\
&\quad - 2\,c_0\,(c_0^{1-\gamma} + \|\mathbf{C}\|_c^2)\,N^{-1} + 2\,c_0\\
&= c_0\,N^{-2}\Big(-\Big(c_0^{1-\gamma} + \|\mathbf{C}\|_c^2\Big)^2\,N^{-1}\,(1+\gamma)\,\langle\,\mathbf{C}^{-1}\,(\mathbf{P_c C})^2,\ \mathbf{b}\rangle_c\\
&\quad +\Big(c_0^{1-\gamma} + \|\mathbf{C}\|_c^2\Big)\gamma\,\|\mathbf{P_c C}\|_c^2\ -\ 2\,N\,(N + \|\mathbf{P_c C}\|_c^2) + 2\,N^2\Big)\\
&= c_0\,N^{-2}\Big(-U^2\,N^{-1}\,(1+\gamma)\,\langle\,\mathbf{C}^{-1}\,(\mathbf{P_c C})^2,\ \mathbf{b}\rangle_c\\
&\quad + U\gamma\,\|\mathbf{P_c C}\|_c^2\ -\ 2\,N\,(N + \|\mathbf{P_c C}\|_c^2) + 2\,N^2\Big)\\
&= \ c_0\,N^{-2}\Big(-U^2\,N^{-1}\,(1+\gamma)\,\langle\,\mathbf{C}^{-1}\,(\mathbf{P_c C})^2,\ \mathbf{b}\rangle_c\\
&\quad + U(\gamma-2)\,\|\mathbf{P_c C}\|_c^2 + 2\,\|\mathbf{P_c C}\|_c^4\Big),
\end{aligned}$$

where

$$U = c_0^{1-\gamma} + \|\mathbf{C}\|_c^2 = Z_0^\gamma\,W^{1-\gamma}\ =\ \|\mathbf{P_c}\,\mathbf{C}\|_c^2 + N. \tag{6.13}$$

Now, for any function $f(x)$,

$$\left(\left(f\left(x^{1/(\gamma-1)}\right)\right)^{1-\gamma}\right)' = (1-\gamma)\,f^{-\gamma}\,f'\,(\gamma-1)^{-1}\,x^{\frac{2-\gamma}{\gamma-1}} = -\,f^{-\gamma}\,f'\,x^{\frac{2-\gamma}{\gamma-1}}, \tag{6.14}$$

and

$$\begin{aligned}
&\Big((f(x^{1/(\gamma-1)}))^{1-\gamma}\Big)''\\
&= \ \gamma\,(\gamma-1)^{-1}\,f^{-\gamma-1}\,(f')^2\,x^{2\,\frac{2-\gamma}{\gamma-1}} - (\gamma-1)^{-1}\,f^{-\gamma}\,f''\,x^{2\,\frac{2-\gamma}{\gamma-1}}\ -\ \frac{2-\gamma}{\gamma-1}\,f^{-\gamma}\,f'\,x^{\frac{3-2\gamma}{\gamma-1}}\\
&= \ x^{\frac{3-2\gamma}{\gamma-1}}\,(\gamma-1)^{-1}\,f^{-\gamma-1}\,\Big(\gamma(f')^2\,x^{1/(\gamma-1)} - f\,f''\,x^{1/(\gamma-1)}\ -\ (2-\gamma)\,f\,f'\Big).
\end{aligned} \tag{6.15}$$

Thus, it remains to show that

$$\gamma\,(f'(W))^2\,W - f(W)\,f''(W)\,W - (2-\gamma)\,f(W)\,f'(W) \ge 0 \tag{6.16}$$

for $f(W) = k(W)/W$ and $W = V^{1/(\gamma-1)}$. By the above (see, also (6.13)),

$$\begin{aligned}
\gamma\ (f'(W))^2\,W\ -\ & f(W)\,f''(W)\,W - (2-\gamma)\,f(W)\,f'(W) \\
&= \gamma\,(W^{-2}\,c_0\,\|\mathbf{P_c}\,\mathbf{c}\|_c^2\,N^{-1})^2\,W \\
&\quad -\ c_0\,W^{-1}\,c_0\,N^{-2}\Big(-U^2\,N^{-1}\,(1+\gamma)\,\langle \mathbf{c}^{-1}\,(\mathbf{P_c c})^2,\ \mathbf{b}\rangle_c \\
&\quad + U(\gamma-2)\,\|\mathbf{P_c c}\|_c^2 + 2\,\|\mathbf{P_c c}\|_c^4\Big)\,W^{-2} \\
&\quad -\ (2-\gamma)\,c_0\,W^{-1}\,W^{-2}\,c_0\,\|\mathbf{P_c}\,\mathbf{c}\|_c^2\,N^{-1} \\
&= c_0^2\,N^{-3}\,W^{-3}\,U^2\,(1+\gamma)\,\langle \mathbf{c}^{-1}\,(\mathbf{P_c c})^2,\ \mathbf{b}\rangle_c \geq 0,
\end{aligned}$$

because $\mathbf{b} \geq 0$ by Lemma 5.7. Identity (6.7) follows from Lemma 7.6. □

Proof of Theorem 3.4 By Proposition 6.3, $g(v)$ is convex and therefore

$$g(v) \geq g(0) + g'(0)\,v \tag{6.17}$$

for all $v \geq 0$. That is, by definition of $g(v)$,

$$\begin{aligned}
\left(\frac{c_0}{W}\right)^{1-\gamma} &\geq \left(c_0^{(1)}\right)^{1-\gamma} + (1-\gamma)\left(c_0^{(1)}\right)^{-\gamma} c_0^{(\gamma)}\,W^{\gamma-1} \\
&= \left(c_0^{(1)}\right)^{1-\gamma} - B(\mathbf{Y})\,W^{\gamma-1}
\end{aligned}$$

with $B(\mathbf{Y}) = (1-\gamma)\left(c_0^{(1)}\right)^{-\gamma} c_0^{(\gamma)}$. Consequently, by Lemma 7.8,

$$\begin{aligned}
\frac{\partial \pi_0(W,\ \mathbf{Y})}{\partial W} &= -1 + \left(\frac{c_0(W)}{W}\right)^{\gamma} Z_0^{\gamma} \\
&\leq -1 + \left(\left(c_0^{(1)}\right)^{1-\gamma} - B(\mathbf{Y})\,W^{\gamma-1}\right)^{\frac{\gamma}{1-\gamma}} Z_0^{\gamma}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\pi_0(W,\ \mathbf{Y}) - \mathbf{Y}_0^u &= \pi_0(W,\ \mathbf{Y}) - \pi_0(0,\ \mathbf{Y}) \\
&\leq -W + Z_0^{\gamma} \int_0^W \left(\left(c_0^{(1)}\right)^{1-\gamma} - B(\mathbf{Y})\,w^{\gamma-1}\right)^{\frac{\gamma}{1-\gamma}} dw \\
&= -W + W\,(c_0^{(1)})^{\gamma}\left(1 - \left(c_0^{(1)}\right)^{\gamma-1} B(\mathbf{Y})\,W^{\gamma-1}\right)^{\frac{1}{1-\gamma}} Z_0^{\gamma}.
\end{aligned}$$

□

Appendix C: Proof of Theorem 3.3

We will need the following auxiliary

Lemma 7.1 *Let* $A,\ B,\ M > 0$ *and* Z *be random variables. The unique random variable* X*, solving*

$$A\,X + B\,E\,[M\,X\,|\,\mathcal{G}_t] = Z \tag{7.1}$$

is given by

$$X = A^{-1}\left(Z - B\,\frac{E\,[M\,A^{-1}\,Z\,|\,\mathcal{G}_t]}{1 + E\,[M\,A^{-1}\,B\,|\,\mathcal{G}_t]}\right). \tag{7.2}$$

In particular,

$$X = \frac{A^{-1}\,Z}{1 + E\,[M\,A^{-1}\,B\,|\,\mathcal{G}_t]} \tag{7.3}$$

if both Z and B are $\mathcal{G}_t$-measurable.

Proof Multiplying both sides of (7.1) by $M\,A^{-1}$ and taking the conditional expectation $P_{\mathcal{G}_t}$, we obtain the expression for $E[M\,X\,|\,\mathcal{G}_t]$. Plugging this into (7.1) gives the required solution. □

Definition 7.2 Let $s_t \in \mathcal{G}_t$ be the event $c_t(0) = 0$. Let also χ_{s_t} be the indicator of the event s_t.

Note that, by (3.7), $s_0 = \Omega$.

First order conditions (2.28) imply that $e^{-\rho}\,E[c_{t+1}^{-\gamma}\,|\,\mathcal{H}_{t+1}]$ and $c_t^{-\gamma}$ are finite or infinite simultaneously. Consequently, the following is true:

Lemma 7.3 $s_{t+1}\,\subset\,s_t$ *for all* $t = 0, \ldots, T-1$.

Note again that we work on a finite probability space.

In Lemma 3.2, we have calculated the limit of the indifference prices as the capital W goes to zero. The next step is to calculate the expansion of the indifference price around $W = 0$.

Let $S_T = 0$. We define the following random variables S_t inductively:

- if $c_t(0)\,\neq\,0$, let

$$K_t = E\left[c_{t+1}(0)^{-\gamma-1}\left(1 - E\left[\frac{M_{t+2}}{M_{t+1}}\,S_{t+1}\,\middle|\,\mathcal{G}_{t+1}\right]\right)\,\middle|\,\mathcal{H}_{t+1}\right] \tag{7.4}$$

and

$$S_t = \frac{e^{\rho}\,K_t^{-1}\,\frac{M_{t+1}}{M_t}\,c_t(0)^{-\gamma-1}}{1 + E\left[K_t^{-1}\,e^{\rho}\,\left(\frac{M_{t+1}}{M_t}\right)^2\,c_t(0)^{-\gamma-1}\,\middle|\,\mathcal{G}_t\right]}; \tag{7.5}$$

- if $c_t(0) = 0$, let

$$K_t = E\left[\left(1 - E_{t+1}\left[\frac{M_{t+2}}{M_{t+1}}\,S_{t+1}\,\middle|\,\mathcal{G}_{t+1}\right]\right)^{-\gamma}\,\chi_{s_{t+1}}\,\middle|\,\mathcal{H}_{t+1}\right] \tag{7.6}$$

and, for $t \geq 1$, let

$$S_t = \frac{e^{-\rho\,\gamma^{-1}}\,K_t^{\gamma^{-1}}\,\left(\frac{M_{t+1}}{M_t}\right)^{-\gamma^{-1}}}{1 + E\left[e^{-\rho\,\gamma^{-1}}\,K_t^{\gamma^{-1}}\,\left(\frac{M_{t+1}}{M_t}\right)^{1-\gamma^{-1}}\,\middle|\,\mathcal{G}_t\right]}; \tag{7.7}$$

- for $t = 0$,

$$S_0 = e^{-\rho\,\gamma^{-1}}\,K_0^{\gamma^{-1}}\,M_1^{-\gamma^{-1}}. \tag{7.8}$$

Let further

$$S_t^{(\Pi)} = \prod_{\tau=0}^{t-1} S_\tau \tag{7.9}$$

and

$$C_t^{(\Pi)} = S_t^{(\Pi)} - E\left[\frac{M_{t+1}}{M_t}\, S_{t+1}^{(\Pi)} \,|\, \mathcal{G}_t\right]. \tag{7.10}$$

Define for $\gamma > 1$

$$c_0^{(1)} = \left(\frac{Z_0^\gamma}{\sum_{t=0}^T e^{-\rho\, t}\, E\left[\left(C_t^{(\Pi)}\right)^{1-\gamma}\ \chi_{s_t}\right]}\right)^{1/(1-\gamma)}. \tag{7.11}$$

Let now, for each $t = 1, \ldots, T$,

$$X_t^{(1)} = S_t^{\Pi}\, c_0^{(1)}.$$

Note that, by definition, $X_t^{(1)} = S_{t-1}\, X_{t-1}^{(1)}$.

Substitute the Ansatz $X_t(W) = X_t(0) + X_t^{(1)}\, W + o(W)$ and doing some routine calculations, we arrive at

Lemma 7.4 *Let $\gamma > 1$. Then,*

$$X_t(W) = \mathbf{Y}_t^u + X_t^{(1)}\, W + o(W) \tag{7.12}$$

and, consequently,

$$c_t(W) = c_t(0) + c_t^{(1)}\, W + o(W) \tag{7.13}$$

with

$$c_t^{(1)} = X_t^{(1)} - E[M_{t+1}\, X_{t+1}^{(1)}\, M_t^{-1} \,|\, \mathcal{G}_t] \tag{7.14}$$

for all $t = 1, \ldots, T$, and $c_0^{(1)}$ is given by (7.11).

Let now

$$\log(c_0^{(\alpha)}) = \left(\sum_{t=0}^T e^{-\rho\, t}\, \mathrm{Prob}[s_t]\right)^{-1} \left(\sum_{t=1}^T e^{-\rho\, t}\, \log\left(\frac{e^{-\rho\, t}\, M_t}{\sum_{t=0}^T e^{-\rho\, t}}\right)\right.$$
$$\left. - \sum_{t=1}^T e^{-\rho\, t}\, \left(E\left[\log c_t(0)\,(1-\chi_{s_t})\right] - E\left[\log C_t^{(\Pi)}\, \chi_{s_t}\right]\right)\right) \tag{7.15}$$

and define for each $t = 1, \ldots, T$,

$$X_t^{(\alpha)} = S_t^{(\Pi)}\, c_0^{(\alpha)}.$$

Substituting the Ansatz

$$c_t = c_t(0) + c_t^{(\alpha)}\, W^\alpha + o(W^\alpha)$$

into the utility indifference equation, we arrive at

Lemma 7.5 *Let $\gamma = 1$ and*

$$\alpha = \alpha(\mathbf{Y}) = \frac{\sum_{t=0}^{T} e^{-\rho t}}{\sum_{t=0}^{T} e^{-\rho t}\,\mathrm{Prob}[s_t]} \; > \; 1. \tag{7.16}$$

Then,

$$X_t = \mathbf{Y}_t^u + W^\alpha\, X_t^{(\alpha)} + o(W^\alpha). \tag{7.17}$$

Consequently,

$$c_t(W) = c_t(0) + c_t^{(\alpha)}\, W^\alpha + o(W^\alpha) \tag{7.18}$$

with

$$c_t^{(\alpha)} = X_t^{(\alpha)} - E\left[\frac{M_{t+1}}{M_t}\, X_{t+1}^{(\alpha)} \,|\, \mathcal{G}_t\right] \tag{7.19}$$

for $t \ge 1$, and $c_0^{(\alpha)}$ is given by (7.15).

We are now ready to calculate the "second" order of the asymptotic expansion. Let $S_T^{(\gamma)} = 0$. We define the random variables $S_t^{(\gamma)}$ inductively:

- if $c_t(0) \neq 0$, let

$$K_{\gamma,\,t} = E\left[c_{t+1}(0)^{-\gamma-1}\left(1 - E\left[\frac{M_{t+2}}{M_{t+1}}\, S_{t+1}^{(\gamma)} \,|\, \mathcal{G}_{t+1}\right]\right) |\, \mathcal{H}_{t+1}\right] \tag{7.20}$$

  and

$$S_t^{(\gamma)} = \frac{e^\rho\, K_{\gamma,\,t}^{-1}\, \frac{M_{t+1}}{M_t}\, c_t(0)^{-\gamma-1}}{1 + E\left[K_{\gamma,\,t}^{-1}\, e^\rho \left(\frac{M_{t+1}}{M_t}\right)^2\, c_t(0)^{-\gamma-1} \,|\, \mathcal{G}_t\right]}; \tag{7.21}$$

- if $c_t(0) = 0$ and $t \ge 1$, let

$$K_{\gamma,\,t} = E\left[\left(c_{t+1}^{(1)}\right)^{-\gamma-1}\left(1 - E_{t+1}\left[\frac{M_{t+2}}{M_{t+1}}\, S_{t+1}^{(\gamma)} \,|\, \mathcal{G}_{t+1}\right]\right) \chi_{s_{t+1}} \,|\, \mathcal{H}_{t+1}\right] \tag{7.22}$$

  and

$$S_t^{(\gamma)} = \frac{e^\rho\, K_{\gamma,\,t}^{-1}\, \frac{M_{t+1}}{M_t}\, \left(c_t^{(1)}\right)^{-\gamma}}{1 + E\left[K_{\gamma,\,t}^{-1}\, e^\rho \left(\frac{M_{t+1}}{M_t}\right)^2\, \left(c_t^{(1)}\right)^{-\gamma} \,|\, \mathcal{G}_t\right]}; \tag{7.23}$$

- for $t = 0$

$$S_0^{(\gamma)} = e^\rho\, K_{\gamma,\,0}^{-1}\, \frac{M_{t+1}}{M_t}\, \left(c_0^{(1)}\right)^{-\gamma-1}. \tag{7.24}$$

Let now for each $t = 1, \ldots, T$

$$S_{t,\,\gamma}^{(\Pi)} = \prod_{\tau=0}^{t-1} S_\tau^{(\gamma)} \quad \text{and} \quad C_{t,\,\gamma}^{(\Pi)} = S_{t,\,\gamma}^{(\Pi)} - E\left[\frac{M_{t+1}}{M_t}\, S_{t+1,\,\gamma}^{(\Pi)} \,|\, \mathcal{G}_t\right]. \tag{7.25}$$

Let also for $\gamma > 1$

$$c_0^{(\gamma)} = \frac{(\gamma - 1)^{-1} \sum_{t=1}^{T} e^{-\rho t} E\left[(c_t(0))^{1-\gamma} (1 - \chi_{s_t})\right]}{(c_0^{(1)})^{-\gamma} + \sum_{t=1}^{T} e^{-\rho t} E\left[(c_t^{(1)})^{-\gamma} C_{t,\gamma}^{(\Pi)} \chi_{s_t}\right]} \tag{7.26}$$

and

$$X_t^{\gamma} = S_{t,\gamma}^{(\Pi)} c_0^{(\gamma)} \tag{7.27}$$

for all $t = 1, \ldots, T$.

Lemma 7.6 *Let $(X_t^{(1)})$ and $(c_t^{(1)})$ be the processes of Lemma 7.4. If $1 < \gamma < 2$, then*

$$X_t(W) = \mathbf{Y}_t^u + X_t^{(1)} W + X_t^{(\gamma)} W^{\gamma} + O(W^2). \tag{7.28}$$

Consequently,

$$c_t = c_t(0) + c_t^{(1)} W + c_t^{(\gamma)} W^{\gamma} + O(W^2) \tag{7.29}$$

with

$$c_t^{(\gamma)} = X_t^{(\gamma)} - E\left[\frac{M_{t+1}}{M_t} X_{t+1}^{(\gamma)} \,|\, \mathcal{G}_t\right]. \tag{7.30}$$

It remains to consider the case $\gamma \geq 2$. We only treat here the case $\gamma > 2$. The case $\gamma = 2$ must be treated separately, because the terms of order W^{γ} will enter the asymptotic expansion. Otherwise, the calculations are almost identical.

Let $S_T^{(2)} = 0$. We define the following random variables $S_t^{(2)}$ inductively:

- if $c_t(0) \neq 0$, let

$$K_{2,t} = E\left[c_{t+1}(0)^{-\gamma-1} \left(1 - E\left[\frac{M_{t+2}}{M_{t+1}} S_{t+1}^{(2)} \,|\, \mathcal{G}_{t+1}\right]\right) \,|\, \mathcal{H}_{t+1}\right] \tag{7.31}$$

  and

$$S_t^{(2)} = \frac{e^{\rho} K_{2,t}^{-1} \frac{M_{t+1}}{M_t} c_t(0)^{-\gamma}}{1 + E\left[K_{2,t}^{-1} e^{\rho} \left(\frac{M_{t+1}}{M_t}\right)^2 c_t(0)^{-\gamma} \,|\, \mathcal{G}_t\right]}. \tag{7.32}$$

  Let also

$$X_{t+1}^{(2)} = S_t^{(2)} X_t^{(2)} + R_t^{(2)} \tag{7.33}$$

  with

$$\begin{aligned} R_t^{(2)} = \; & S_t^{(2)} \frac{1}{2} (\gamma + 1) \\ & \Big(e^{-\rho} c_t(0)^{\gamma+1} M_{t-1} M_t^{-1} E\left[c_{t+1}^{-\gamma-2}(0) \left(c_{t+1}^{(1)}\right)^2 \,|\, \mathcal{H}_{t+1}\right] \\ & - c_t^{-1}(0) \left(c_t^{(1)}\right)^2 \Big)\Big); \end{aligned} \tag{7.34}$$

- if $c_t(0) = 0$, let

$$K_{2,t} = E\left[(c_{t+1}^{(1)})^{-\gamma-1} \left(1 - E_{t+1}\left[\frac{M_{t+2}}{M_{t+1}} S_{t+1}^{(2)} \,|\, \mathcal{G}_{t+1}\right]\right) \chi_{s_{t+1}} \,|\, \mathcal{H}_{t+1}\right] \tag{7.35}$$

and

$$S_t^{(2)} = \frac{e^{\rho}\, K_{2,\,t}^{-1}\, \frac{M_{t+1}}{M_t}\, \left(c_t^{(1)}(0)\right)^{-\gamma}}{1 + E\left[K_{2,\,t}^{-1}\, e^{\rho}\, \left(\frac{M_{t+1}}{M_t}\right)^2\, \left(c_t^{(1)}(0)\right)^{-\gamma}\,\middle|\, \mathcal{G}_t\right]}. \tag{7.36}$$

Then, for $t \geq 1$,

$$X_{t+1}^{(2)} = S_t^{(2)}\, X_t^{(2)}; \tag{7.37}$$

- for $t = 0$

$$S_0^{(2)} = e^{\rho}\, K_{2,\,0}^{-1}\, \frac{M_{t+1}}{M_t}\, (c_0^{(1)})^{-\gamma-1} \tag{7.38}$$

and

$$X_1^{(2)} = S_0^{(2)}\, c_0^{(\gamma)}. \tag{7.39}$$

Let now for each $t = 1, \ldots, T$,

$$S_{t,\,2}^{(\Pi)} = \prod_{\tau=0}^{t} S_\tau^{(2)}$$

and

$$S_{t,\,2}^{(Q)} = S_{t,\,2}^{(\Pi)} \sum_{\tau=1}^{t} R_\tau\, \left(S_{\tau,\,2}^{(\Pi)}\right)^{-1}.$$

Then,

$$X_t^{(2)} = S_{t,\,2}^{(\Pi)}\, c_0^{(\gamma)} + S_{t,\,2}^{(Q)}. \tag{7.40}$$

Let also

$$\begin{aligned} C_t^{(\Pi)} &= S_{t,\,2}^{(\Pi)} - E\left[\frac{M_{t+1}}{M_t}\, S_{t+1,\,2}^{(\Pi)} \,|\, \mathcal{G}_t\right], \\ C_t^{(Q)} &= S_{t,\,2}^{(Q)} - E\left[\frac{M_{t+1}}{M_t}\, S_{t+1,\,2}^{(Q)} \,|\, \mathcal{G}_t\right], \end{aligned}$$

and

$$c_0^{(2)} = -\frac{\sum_{t=1}^{T} e^{-\rho\, t}\, E\left[\left(c_t^{(1)}\right)^{-\gamma}\, C_t^{(Q)}\, \chi_{s_t}\right]}{(c_0^{(1)})^{-\gamma} + \sum_{t=1}^{T}\, E\left[\left(c_t^{(1)}\right)^{-\gamma}\, C_t^{(\Pi)}\, \chi_{s_t}\right]}. \tag{7.41}$$

Lemma 7.7 *Let $(X_t^{(1)})$ and $(c_t^{(1)})$ be the processes, constructed in Lemma 7.4. If $\gamma > 2$, then*

$$X_t(W) = \mathbf{Y}_t^u + X_t^{(1)}\, W + X_t^{(2)}\, W^2 + O(W^{\min\{3,\,\gamma\}}). \tag{7.42}$$

Consequently,

$$c_t = c_t(0) + c_t^{(1)}\, W + c_t^{(2)}\, W^2 + O(W^{\min\{3,\,\gamma\}}) \tag{7.43}$$

with

$$c_t^{(2)} = X_t^{(2)} - E\left[\frac{M_{t+1}}{M_t}\, X_{t+1}^{(2)} \,\middle|\, \mathcal{G}_t\right] \tag{7.44}$$

for $t \geq 1$.

Now, with the expansion for the optimal consumption stream on our hands, we can calculate the expansion for the premium.

We will need the following important identity (see, Corollary 4.11).

Lemma 7.8 *We have*

$$\frac{\partial \pi_0(W,\, \mathbf{Y})}{\partial W} = -1 + \left(\frac{c_0(W)}{W}\right)^{\gamma} \, Z_0^{\gamma} \tag{7.45}$$

for $\gamma > 1$, *and*

$$\frac{\partial \pi_0(W,\, \mathbf{Y})}{\partial W} = -1 + \frac{c_0(W)}{W} \, (1 + \langle\, \mathfrak{d},\, \mathbf{1} \rangle) \tag{7.46}$$

for $\gamma = 1$.

Proof By (5.12), (2.32) and (4.39),

$$\begin{aligned} 0 =& \; (c_0 - \pi_0 - W)\, c_0^{-\gamma} + \langle\, \mathbf{c} + \mathbf{Y},\, \mathfrak{d}\, \mathbf{c}^{-\gamma}\, \rangle \\ =& \; c_0^{1-\gamma} + \langle\, \mathbf{c},\, \mathfrak{d}\, \mathbf{c}^{-\gamma}\, \rangle - (\pi_0 + W)\, c_0^{-\gamma} + \langle\, \mathbf{Y},\, \mathfrak{d}\, \mathbf{c}^{-\gamma}\, \rangle \\ =& \; W^{1-\gamma}\, Z_0^{\gamma} - (\pi_0 + W)\, c_0^{-\gamma} + \langle\, \mathbf{Y},\, \mathfrak{d}\, \mathbf{c}^{-\gamma}\, \rangle. \end{aligned}$$

It is now not difficult to see that

$$c_0^{-\gamma}\, \frac{\partial \pi_0(\mathbf{Y})}{\partial \mathbf{Y}}(\mathbf{Y}) = \langle\, \mathbf{Y},\, \mathfrak{d}\, \mathbf{c}^{-\gamma}\, \rangle = (W + \pi_0)\, c_0^{-\gamma} - W^{1-\gamma}\, Z_0^{\gamma}. \tag{7.47}$$

Identity (7.45) follows now from homogeneity of π_0 (see, Proposition 3.1). If $\gamma = 1$, identity (5.12) takes the form

$$0 = (c_0 - \pi_0 - W)\, c_0^{-1} + \langle\, \mathbf{c} + \mathbf{Y},\, \mathfrak{d}\, \mathbf{c}^{-1}\, \rangle$$

and therefore

$$c_0^{-1}\, \frac{\partial \pi_0(\mathbf{Y})}{\partial \mathbf{Y}}(\mathbf{Y}) = \langle\, \mathbf{Y},\, \mathfrak{d}\, \mathbf{c}^{-1}\, \rangle = (\pi_0 + W)\, c_0^{-1} - (1 + \langle\, \mathfrak{d},\, \mathbf{1} \rangle)$$

and (7.46) follows again from homogeneity of π_0. □

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3 Let first $\gamma = 1$. By Lemma 7.5, $c_0(W) = c_0^{(\alpha)}\, W^{\alpha} + o(W^{\alpha})$. Integrating (7.46) with respect to W, we get

$$\pi_0(W,\, \mathbf{Y}) = \mathbf{Y}_0^u - W + \alpha^{-1}\, c_0^{(\alpha)}\, (1 + \langle\, \mathfrak{d},\, \mathbf{1} \rangle)\, W^{\alpha} \;+\; o(W^{\alpha}) \tag{7.48}$$

which is what had to be proved.

Let now $\gamma \in (1, 2)$. Then,

$$\frac{c_0(W)}{W} = c_0^{(1)} + W^{\gamma-1}\, c_0^{(1)} + O(W)$$

and therefore

$$\left(\frac{c_0(W)}{W}\right)^{\gamma} = \left(c_0^{(1)}\right)^{\gamma} + \gamma\, \left(c_0^{(1)}\right)^{\gamma-1}\, W^{\gamma-1}\, c_0^{(1)} \;+\; O(W).$$

Integrating (7.45) with respect to W, we get the required.

Other expansions follow in the same manner from Lemma 7.7 and (7.45). □

Appendix D: Small claims/capital ratio

In this section we study the asymptotic behavior of the premium π when the size of the claims $\mathbf{Y}$ is small relative to the capital of the company. Since we have an explicit formula for the consumption stream when there are no claims (see, Proposition 4.10), the derivative $\mathbf{P_c}$ can be calculated explicitly by a recursive procedure. But, the expression is rather complicated. For the readers convenience, we perform the calculation for the so-called idiosyncratically incomplete markets. It is characterized in the following definition.

Definition 8.1 A market $(\mathcal{M}, \mathcal{G})$ is idiosyncratically incomplete if

(1) There exists a subfiltration $\mathcal{F} = (\mathcal{F}_t)$ of $\mathcal{G}$ with $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t = 0, \ldots, T$ such that the price and dividend process of any asset in the market is adapted to $\mathcal{F}$;
(2) The market $\mathcal{M}$ is complete with respect to $\mathcal{F}$ (but not with respect to $\mathcal{G}$). That is, any $\mathcal{F}$-adapted process can be replicated by an $\mathcal{F}$-adapted portfolio strategy;
(3) Filtration $\mathcal{G}$ does not contain any additional information about events in $\mathcal{F}$. Formally,

$$E[X \mid \mathcal{F}_t] = E[X \mid \mathcal{G}_t] \tag{8.1}$$

for any $\mathcal{F}_{t+1}$-measurable variable X.

Any idiosyncratically incomplete market belongs to the class $\mathfrak{C}$ with the hedgeable algebra given by $\mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{G}_{t-1})$, the minimal algebra, containing $\mathcal{F}_t$ and $\mathcal{G}_{t-1}$, and the aggregate state price density process $\mathbf{M}$ is, in fact, $\mathcal{F}$-adapted.

Definition 8.1 means that, without a random endowment, an agent faces a complete market and his optimal consumption stream is given by the standard, complete market formula (see, (4.1) and (4.2))

$$c_t = e^{-\rho t \gamma^{-1}} M_t^{-\gamma^{-1}} c_0 \tag{8.2}$$

and

$$c_0 = \frac{W}{Z_0} = \frac{W}{1 + \sum_{t=1}^{T} e^{-\rho t \gamma^{-1}} E\left[M_t^{1-\gamma^{-1}}\right]}. \tag{8.3}$$

Furthermore, the process Z_t takes the following simple form,

$$Z_t = \mathrm{P}_{\mathcal{F}_t} \sum_{\tau=t}^{T} e^{-\rho t \gamma^{-1}} M_\tau^{1-\gamma^{-1}}.$$

See, Proposition 4.10.

Definition 8.2 For each $t = 1, \ldots, T$, let

$$\mathrm{I}_t(\mathbf{y}, \mathbf{M}) = E\left[\sum_{\tau=t}^{T} y_\tau M_\tau \,\middle|\, \mathcal{G}_t\right] - E\left[\sum_{\tau=t}^{T} y_\tau M_\tau \,\middle|\, \mathcal{H}_t\right]. \tag{8.4}$$

We can now calculate the second order approximation to the indifference price when the ratio $\mathbf{Y}/W$ of claims to capital is small. A direct (but tedious) calculation implies the following

Proposition 8.3 *We have*

$$\pi_0(W,\ \mathbf{Y}) = W\,\pi_0(1, \mathbf{Y}/W) \tag{8.5}$$

and therefore, when $\mathbf{Y}/W$ *is small,*

$$\pi_0(W,\ \mathbf{Y}) = \langle \mathbf{Y},\ \mathbf{M}\rangle + \ W\,Z_0 \sum_{t=1}^{T} E\left[\frac{\mathrm{Var}_{\mathcal{F}_t}(\mathrm{I}_t(\mathbf{Y})\,W^{-1})}{Z_t}\right] + W\,O((\mathbf{Y}/W)^3). \tag{8.6}$$

The expansion of Proposition 8.3 can be viewed as an analog of the result of Kramkov and Sirbu [27] with intermediate consumption and a finite probability space. Of course, in our setting, calculating the expansion is just an exercise in computing expectations. Proving Proposition 8.3 in the general semi-martingale setting of Kramkov and Sirbu would require substantial technical difficulties to be resolved. We also mention a recent paper of Kramkov and Sirbu [28] where the optimal hedging strategy for the untraded claim is explicitly calculated.

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